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On the electromagnetic behaviour of a plasma

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With 31 figures in the text

ABSTRACT

A plasma resonator experiment gives several resonances, whereas theory predicts only one for a homogeneous zero-temperature plasma cylinder of circular cross section. A theory is presented that predicts: that also an inhomogeneous plasma cylinder generally has only one dipole resonance; that a deformation of the cross section or introduction of a longitudinal magnetic field splits the resonance into two; and that an elementary model of a plasma of non-zero temperature gives an infinite series of resonances. This theory agrees qualitatively with the experimental results.

The surface on which the dielectric constant vanishes in an inhomogeneous plasma constitutes a mathematical singularity. A suitable treatment is presented that resolves the paradox of losses in a plasma where, by assumption, there are no collisions.

1. Introduction

As early as 1931 Tonks [1] put a low pressure mercury discharge tube of circular cross section between two condenser plates loading a Lecher line. By varying the current density in the tube he found one strong resonance and a weaker one. Several years later Shapiro [2] reported measurements of a similar character. He, however, found only one resonance. These two experiments were done at a fairly low frequency, a few hundred Mc/s.

In connection with his studies of the scattering of electromagnetic waves from meteor trails, Herlofson [3] realized the value of laboratory experiments. In 1951 Romell [4] thus published results of some measurements made at this laboratory on the scattering of electromagnetic waves from a low pressure mercury discharge tube. The electric vector was perpendicular to the axis of the tube. The frequency he used was 10^9 c/s, i.e. a frequency a few times greater than that used in the two experiments just mentioned. Romell reported three resonances with decreasing amplitude for decreasing plasma density.

Then, in 1957, Dattner [5] published the results of measurements on the energy transmitted through a wave guide containing a discharge tube. The electric field in the wave guide was perpendicular to the axis of the tube. Dattner observed minima in transmitted energy at approximately the same densities as those at which Romell had observed maxima in reflected energy. The frequency was five times that used by Romell, and the diameter of the discharge tube was decreased in the same ratio. This should make the effect of the tube the same as in Romell's case.

Romell measured the space field distribution round the tube and found that the scattered field was a dipole field. Several years later Boley [6] repeated the same measurement and came to the same conclusion for the two major resonances. Even if the measurements are conclusive only for the major resonances, it seems quite plausible that all the observed resonances have dipole space distribution.

Tonks reported measurements with a discharge tube of elliptic cross section. From his curves one finds that there is one resonance corresponding to each of the axes of the cross section (Tonk's conclusion is somewhat different). In Dattner's experiment we also find that each single resonance for circular cross section splits into two resonances for elliptic cross section—corresponding to the two axes of the ellipse.

In simple resonance systems we generally find two kinds of energy which we may call potential energy and kinetic energy. A plasma cylinder is no exception. In the simplest case, which we shall deal with here, the plasma density is not a function of the coordinate along the axis of the cylinder. Let us begin with a cylinder where the space charge density is zero everywhere and all particles are at rest. If we then perturb the position of the electrons, an electric field is created and this field tends to decrease the perturbation. The electrons are set into motion. As usual, when the perturbation in position has become zero, the particles have a non-zero velocity and swing out in the direction opposite to the original perturbation. Again we find that a field is created that tends to diminish the perturbation. The system is oscillatory. The two kinds of energy we find here are electric field energy, which has to be regarded as potential energy, and kinetic energy of the electrons. The positive ions and possible negative ions present are so heavy that, near the resonance frequency, their motion can be neglected and consequently also their contribution to the kinetic energy. Since the energy of the system is periodically transformed from kinetic energy into potential energy and back again, the condition for resonance is that the total energy is independent of time. Since the potential energy is not restricted to the interior of the cylinder, the resonance frequency generally depends not only on the conditions inside the plasma region, but also on the shape of the plasma region and the conditions outside.

For every particle in the medium we have the equation of motion

$$i\,\omega\,m\,\mathbf{v} = e\,\mathbf{E},$$

where $\mathbf{v}$ is the velocity of the particle, e and m its charge and mass, and $\mathbf{E}$ the electric field. If the perturbation is periodic with the angular frequency ω, we then easily find that the kinetic energy per unit volume is

$$T = \tfrac{1}{2}\,n\,m\,v^2 = \frac{n\,e^2/m\,\varepsilon_0}{\omega^2}\,\frac{\varepsilon_0\,E^2}{2},$$

where n is the electron density. If we denote the electric field energy per unit volume by W, we find that the dielectric constant, as introduced by Bergmann and Düring 1929 [7], can also be written

$$\varepsilon = \frac{W - T}{W}.$$

We stress that T and W are time averages of the energy densities and not of the total values for the whole system.

Thus, in the case of a homogeneous medium with no oscillating electric field outside, we would expect resonance when $\varepsilon = 0$: a relation given by J. J. Thomson long ago [8].

Since generally there are regions which do not contain plasma, i.e. regions where the kinetic energy is zero, we must also have regions where the kinetic energy exceeds the potential energy, as equality between total kinetic energy and total potential energy is necessary for resonance. This means that there must be regions where ε is negative.

Actually we find that, for resonance, the integral of

$$\frac{\mathbf{E}\,\mathbf{D}}{2}$$

extended over the whole space must vanish. (At this point it is perhaps necessary to stress that, in systems where the fields are independent of time, the above quantity represents the total potential energy of the system. For time-dependent fields this is approximately correct at frequencies far below the resonant frequencies of the system; but near resonance the kinetic energy of the system must be taken into account and then the integral is no longer equal to the energy of the system but to the difference between the potential and kinetic energy.)

In the case of a cylinder of uniform density the electric field does not vanish outside the cylinder. Since the kinetic and potential energy must be equal at resonance we find that ε for the plasma region must be negative. In this special case the electric field energy is equally divided between the inner and the outer region if we assume static field distribution; we find that resonance occurs when $\varepsilon = -1$, a condition given by Tonks [1].

Up to now we have assumed that the only two kinds of energy of interest are the kinetic energy of the electrons in the plasma region and the electrostatic field energy. This means that we have neglected the magnetic field energy, which is acceptable if the extension of the plasma region is small enough. We have also neglected the thermal motion in the plasma. With these limitations we have found that we cannot expect any resonance effects if the dielectric constant is positive everywhere in space.

As ε also has the ordinary meaning, the field distribution can be determined in the usual way. For any volume consisting of a finite number of homogeneous regions we will always find finite values for the field quantities—also in the case where $\varepsilon = 0$ in some region or regions. In particular, we may mention that any region enclosed by a region with $\varepsilon = 0$ is completely shielded and that the potential is constant on the surface of the shielded region. This means that the normal value of the electric field strength at the inner surface may increase indefinitely with decreasing thickness of the enclosing shell. This discussion indicates that the surface $\varepsilon = 0$ in an inhomogeneous medium may be a singular surface, on which the electric field strength is infinite. Since infinities do not exist in nature there must be some unphysical assumption in our reasoning. Let us therefore for a moment study the concept of the dielectric constant of a plasma.

True continuous space charge does not exist in reality since charge is quantized into particles. For a plasma we thus ought to introduce the charges as singularities in vacuum. Because of the enormous number of singularities it is usually convenient to average the field quantities over regions that are large compared with the interparticle distances and in that way be able to treat the plasma as a continuous medium. This point of view led to the concept of the dielectric constant of a plasma. With reference to Lorentz's work [9] on the dielectric constant of solid and liquid media, Hartree [10] criticized the derivation of the plasma dielectric constant on the ground that the Lorentz polarization term was not included [9]. In two important papers Darwin has then proved that the original definition is the correct one in the case of a plasma region [11, 12].

The continuum treatment should be valid when the relative variation of the field quantities within an interparticle distance is insignificant. In regions where ε is different from zero this condition is generally fulfilled but in regions where ε changes sign by passing through zero a careful study is necessary. To exemplify, we can look at a plane condenser where the electric displacement, D, has the same value everywhere between the condenser plates. If ε varies linearly with the coordinate perpendicular to the plates and passes through zero at some plane, the electric field, E, must become infinite as the inverse of the distance to this plane. Then there must be a region near that plane in which the relative variation of the electric field over an interparticle distance is of the order of unity or larger. This clearly shows that a real medium cannot be treated as a continuous medium in the vicinity of a region where ε is zero (cp. [13], [14]).

In the case of an inhomogeneous plasma it is difficult to obtain the solution in a closed form using tabulated functions and to find the solutions by numerical methods would be very laborious. Therefore we may ask whether it is possible to approximate the continuous distribution by consecutive homogeneous regions. To gain some insight into this problem let us for a moment return to the plane condenser and let us assume that the dielectric medium consists of a number of homogeneous plasma regions of different density. The electric displacement, D, must then have the same value in all regions. When the frequency is such that ε is almost zero in one of these regions, the electric field is much larger in that region than in any other of the layers. For a given value of D, the difference between the potential and the kinetic energy in our dielectric medium tends to plus infinity when ε tends to zero from one side and to minus infinity when we approach from the other side. Thus between two consecutive singular values there must be some intermediate frequency for which this difference is zero and we should expect as many resonances as there are layers. This simple example seems to indicate that the density in an inhomogeneous region cannot be approximated by a step function.

Up to this point we have neglected the thermal motion. As a first attempt to investigate the effect of a non-zero temperature it may seem reasonable to introduce the electron pressure in the macroscopic equation of motion. Doing so one finds that, for a given exciting field, the solution is the same as for zero temperature. However, one also gets a spectrum of discrete modes representing oscillations inside the plasma region, which are not coupled to the field in the space outside. If ω is given, there are only a finite number of density value at which these oscillations exist. All these values fall in the region $\varepsilon > 0$.

To treat the effect of the thermal motion by macroscopically introducing the electron pressure is surely a crude method to find the effect of a non-zero temperature. A more satisfactory way would be to study the motion of every separate particle and to sum up the effects of the individual particles. This method, though complicated, is surely the only method that can yield the detailed behaviour of a hot plasma.

The theoretical discussion of the electromagnetic behaviour of a plasma began shortly after the electron was discovered. First one investigated the propagation of electromagnetic waves. Later one became interested in oscillations in plasma regions much smaller than the vacuum wave length. J. J. Thomson (1928) [8] may have been the first to publish an investigation in this field, which was followed by one by Tonks and Langmuir [15]. Three years later, Tonks [1] discussed the influence of the shape of the plasma region.

The first reports were restricted to homogeneous plasma region. In 1931 Tonks also considered an inhomogeneous region in the plane case. He gave only the reactive component of the impedance, however, and thus avoided the difficulty due to the singular surface at $\varepsilon = 0$. In the cylindrical case there are two singularities, one at the surface, $\varepsilon = 0$, and, as is usual in cylindrical problems, one at the origin. Generally, it is impossible to find a solution that is regular at both singularities, and one has to find an artifice which gives an acceptable solution. One then ought to take the solution that is regular at the origin. The other method of removing the other singularity has been to introduce losses in the plasma (Herlofsson, 1951 [3], Kaiser and Closs, 1952 [16]). Having connected the solution on the two sides of the singularity, one lets the collision frequency tend to zero and finds that, even in the limit of zero collision frequency, energy is absorbed. In the case of reflections of plane electromagnetic waves the same phenomenon is found (Budden, 1955 [17]). Budden states that the group velocity decreases with the distance to the singular surface, to become zero there. Hence energy will pile up at this surface. In this way Budden tries to explain the difference between the incident and the reflected energy.

In the case of an inhomogeneous plasma it is difficult to obtain a solution in closed form using tabulated functions. Therefore Makinson and Slade [18] tried to approximate the continuous distribution by a finite number of homogeneous regions. In the case of a homogeneous cylinder there is only one dipole resonance. Makinson and Slade found that, for an increasing number of layers, the number of resonances increase, but not as rapidly as the number of layers. In this way they tried to explain the series of dipole resonances observed by Romell.

W. O. Schumann [19] attempted a different method to explain the many observed dipole resonances. He introduced a plasma region as dielectric medium for a short length of a two-wire line. The plasma was inhomogeneous with a discontinuous drop in density at the boundary. He assumed a non-zero collision frequency. In this way he found, however, only two maxima in the energy passing the loaded part of the line.

At the Linde conference in 1959, Gould [20] presented a paper where he had introduced a non-zero electron pressure in the macroscopic equation of motion. By changing the electron density from zero to infinity, one finds a finite number of resonances. The number of resonances decreases with increasing temperature. Consequently, the relative positions of the resonances depend on the temperature.

Hershberger [21] also assumes a non-zero temperature. He finds an infinite number of maxima and minima in the reflected wave which are due to some kind of interference. In contrast with the result obtained by Gould, the relative positions of his maxima are independent of the temperature and the tube diameter.

The present investigation originated with the wish to explain the many dipole resonances observed by Romell. Obviously a homogeneous cylinder of circular cross section is not a satisfactory model. Merely assuming a different shape of the cross section is not enough. We find only a doubling of the number of resonances for an elliptic cross section and the same result is to be expected for any shape. It thus seems that the experimental results cannot be explained on the model of a cold, homogeneous region.

A cold inhomogeneous plasma region exhibits a singularity at the surface $\varepsilon = 0$, which requires attention. According to the usual method one assumes that the perturbations are so small as to make non-linear effects negligible, that there is no thermal motion, and that there are no collisions—and yet there are losses. That fact ought to make us suspicious. Actually we find that near and at the singular surface the plasma does not behave as a continuous medium. Taking this into account we again investigate the scattering from an inhomogeneous plasma cylinder. We now find only one resonance of dipole character. Further there is no absorption of energy, and the energy radiated at resonance is the same as for a homogeneous cylinder of the same diameter. (G. Boyd at California Institute of Technology has carried out a similar investigation, but he treated the singularity in the common way.) Although we have investigated only a special case our result seems to indicate that an inhomogeneity in the plasma cannot increase the number of resonances over the number we find for a homogeneous plasma region.

We also investigate the possibility of approximating the continuous distribution by a series of homogeneous layers. From a study of a plane condenser with a dielectric material of such a character we find that the continuous density distribution cannot be approximated by a discontinuous one.

Since the attempts to explain the dipole resonances on the model of a cold plasma have failed, we try to take the effect of thermal motion into account. In order to simplify the problem as much as possible, we discuss the plane case with only one value on thermal velocity in the unperturbed case. We then obtain an infinite series of resonances. At the present stage of development, it seems hopeful that we shall be able to explain the experimental results by introducing a non-zero temperature, even if this goal has not yet been reached.

2. Plane geometry

2.1 *Zero temperature*

(a) *Introduction*

We begin with the simplest possible case and step by step introduce new features and discuss the consequences of each step. In order not to stray too far from the main subject we have found it necessary to exclude many interesting questions. For instance we have only touched on the question of the stability of a plasma, which is such a vast subject that it would require a separate investigation.

It is natural to begin with the one dimensional (plane) problem. When we then introduce the inhomogeneous plasma we come across the singularity at $\varepsilon = 0$ and we have to resolve the difficulty thereby introduced. Our discussion shows that the continuity of the density distribution is essential: an attempt to approximate a continuous density distribution by a discontinuous one results in a solution that is not a valied approximation to the solution of the continuous case.

For those familiar with circuit theory the introduction of an equivalent circuit will prove helpful. Therefore, we have in this paper used that possibility, despite the fact that in one case we obtain a capacitance of negative sign.

The plasma can also be thought of as the dielectric medium of a two wire (or two plate) line. The field of research is too large and we have only hinted at this possibility.

Let us introduce two parallel plane plates of infinite extension. The distance between them may be d and the potential difference U. The upper one may be defined to be positive. The medium in between is constituted of free charged particles of some few different species. For each species the density may be a function of the coordinate perpendicular to the plane of the plates but independent of the other cartesian coordinates. We assume that, in the unperturbed state, the particle velocity in the direction perpendicular to the plates shall vanish. Since we shall not assume that we generally have space charge neutrality, we instead assume a homogeneous magnetic field in the direction of the plane of the plates to ensure that an equilibrium is possible in the unperturbed state.

A continuous distribution of space charge never occurs in nature. Space charge is formed by a more or less regular pattern of discrete charges. For our calculations we can, however, with few exceptions, think of the charge as smeared out to form a continuous distribution. As this will simplify the mathematical treatment considerably, we shall in the following assume that we have a continuous distribution of space charge. Since the value of each discrete charge is small and the number of particles per unit volume large, the error introduced is generally very small, but we must ascertain in every specific example that the smearing out is a suitable approximation.

To simplify the treatment we shall also assume that the distance between the plates is much smaller than the vacuum wavelength of the perturbing field.

In order to be able to calculate the perturbation of the space charge we need to know the motion of the individual particles. We then must know the electric field along the path of the particle. This field will be different for the different kinds of particles present at any given position.

The electric field that a particle feels may conveniently be divided into three parts. The source of the first is the applied voltage U between the plates. This field strength equals $-U/d$. Consequently this field has the same value everywhere between the plates.

The source of the second part is the displacement of the charges from their equilibrium position. As we will find the field created by this source has, in the plane case, the same value everywhere between the plates and the field created is independent of the position of its source.

The third source is the relative motion of the different kinds of particles. We here count those particles that pass the level at which the test particle, which may be any of the particles present, is situated. Consequently this part, and this part alone, does depend on the condition of the medium at the point where the particle under study is situated, and it exists only if there are more than one kind of particle present.

For the study of the second part we begin by looking at the space charge due to one species of particles only. We may then add the similar contributions from other kinds of particles. The space charge density we call ϱ. Let us take a layer of thickness δy at the distance y above the lower plate, as shown in Fig. 1. If the voltage between the plates is zero we find that the voltage at the position of the charge, due to the charge $\varrho\,\delta y$, is

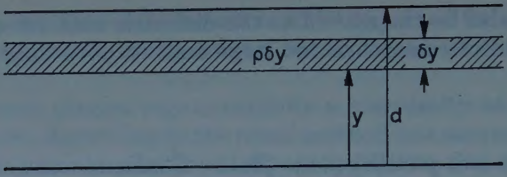

Fig. 1. An infinitesimal plasma layer between parallel metal plates.

$$U = \frac{y\,(d-y)}{d}\,\frac{\varrho\,\delta y}{\varepsilon_0}. \tag{1}$$

Consequently the corresponding electric field is

(*1*) above the charged region

$$E = \frac{y}{d}\,\frac{\varrho\,\delta y}{\varepsilon_0}, \tag{2}$$

(*2*) below the charged region

$$E = -\frac{d-y}{d}\,\frac{\varrho\,\delta y}{\varepsilon_0}. \tag{3}$$

In the present investigation we are not specifically interested in this field but more in the change created by perturbations of the position of space charge. Assume the perturbed position to be $y + \eta(y)$. The perturbation of the field due to the perturbation η in the position we may call δE.

$$\delta E = \frac{\eta}{d}\,\frac{\varrho\,\delta y}{\varepsilon_0}. \tag{4}$$

We thus find that the field δE has the same value everywhere between the plates. We also find that the change is independent of the position of its source.

We now come to the third part. We assume a background charge density ϱ. If our test particle moves upwards the distance Δy relative to the particles constituting the background, then the test particle has passed the charge $\varrho \Delta y$. To begin with this charge layer was above the test particle and the field created was that given by equation (3). After the perturbation is introduced the same charge is below the test particle and the field felt is that given by equation (2). The change in field due to the perturbation of the position of the test particle is thus the difference between the values given by the equations (3) and (2). Let us call the perturbation of the position of the test particle η_μ. In these equations we then have to introduce $\Delta y = \eta_\mu - \eta_\nu$ where μ_ν is the perturbation of the background charges. The field due to this source is thus

$$E = \frac{\varrho_\nu}{\varepsilon_0}\,(\eta_\mu - \eta_\nu). \tag{5}$$

To summarize our result, let us introduce n kinds of particles with densities $\varrho_\nu(y)$ and with perturbed positions $y + \eta_\nu(y)$. The field $E_\mu(y)$ felt by a test particle of the μ^{th} species is then

$$E_\mu = -\frac{U}{d} + \sum_{\nu=1}^{n} \int \frac{\eta_\nu(s)}{d}\,\frac{\varrho_\nu(s)\,d\,s}{\varepsilon_0} + \sum_{\nu=1}^{n} \frac{\varrho_\nu}{\varepsilon_0}\,(\eta_\mu - \eta_\nu), \tag{6}$$

where the integral has to be extended over the whole charged region.

Before we write the equation of motion we shall, when there is a non-vanishing magnetic field, make a Lorentz transformation to a frame that moves with the velocity

$$\frac{\mathbf{E}\times\mathbf{B}}{B^2},$$

where $\mathbf{E}$ is the unperturbed value. We then assume that, in this frame, the particles are at rest when unperturbed. Since the unperturbed electric field is in general a function of the coordinate this transformation has to be made separately for particles at different positions. For any particle we now find

$$m\,\frac{d\,\mathbf{v}}{d\,t} = e\,\mathbf{E} + e\,\mathbf{v}\times\mathbf{B}, \tag{7}$$

where e is the charge with sign. $\mathbf{E}$ is the value given by equation (6) incorporating proper direction. In component form the last equation is

$$\left.\begin{aligned} \frac{d\,v_x}{d\,t} &= v_y\,\frac{e\,B}{m}, \\ \frac{d\,v_y}{d\,t} &= \frac{e\,E_y}{m} - v_x\,\frac{e\,B}{m}, \end{aligned}\right\} \tag{8}$$

if the y coordinate is perpendicular to the plates and if the magnetic field is directed along the z-axis. The first of these equations we can integrate once and obtain

$$v_x = \eta\,\frac{e\,B}{m}. \tag{9}$$

Introducing this into the second equation we find

$$\frac{d^2\eta}{d\,t^2} + \left(\frac{e\,B}{m}\right)^2 \eta = \frac{e\,E}{m}. \tag{10}$$

When we have many kinds of particles present, we get a considerable simplification if we, instead of η, use

$$r_\mu = \frac{m_\mu\,\eta_\mu}{e_\mu}. \tag{11}$$

If we now introduce the equations (6) and (11) into the equation of motion (10) and write it specifically for the μ^{th} kind of particle, we obtain

$$\frac{d^2 r_\mu}{d\,t^2} + \left(\frac{e_\mu B}{m_\mu}\right)^2 r_\mu + \sum_{\nu=1}^{n} \frac{e_\nu\,\varrho_\nu}{m_\nu\,\varepsilon_0}\,r_\nu - \frac{e_\mu\,Q}{m_\mu\,\varepsilon_0}\,r_\mu = \sum_{\nu=1}^{n} \int \frac{r_\nu(s)}{d}\,\frac{e_\nu\,\varrho_\nu(s)}{m_\nu\,\varepsilon_0}\,d\,s - \frac{U}{d}, \tag{12}$$

where
$$Q(y)=\sum_{\nu=1}^{n}\varrho_\nu(y).$$

Thus Q is the net charge density. It vanishes for a neutral medium.

For the special case that the exciting voltage is harmonic in time, we may introduce the admittance Y per unit area of the plates, since we have assumed the distance between the plates to be much smaller than the vacuum wavelength for the exciting frequency. The charge density on the upper plate is equal to the electric displacement at the surface, with opposite sign. The current is the time derivative of that charge density. Hence the admittance is

$$Y=i\,\frac{\omega\,\varepsilon_0}{d}-i\,\frac{\omega\,\varepsilon_0}{d}\left(\sum_{\nu=1}^{n}\int\eta_\nu\,\varrho_\nu\,d\,s\right)(\varepsilon_0\,U)^{-1}, \tag{13}$$

where the first term represents the vacuum capacitance between the plates. The motion of the space charge thus represents an admittance in parallel with this vacuum capacity.

If we see the structure as a parallel line for propagation and restrict the discussion to those cases where the wavelength is much larger than the distance between the plates, we find, as usual, that the phase velocity v_p is given by

$$\frac{v_p}{c}=\left(\frac{i\,\omega}{\mu_0\,d\,Y}\right)^{\frac{1}{2}},$$

where c is the velocity of light in vacuum.

(b) *One-species medium (electrons)*

In this case we can drop the index. Equation (12) is then simplified to

$$\frac{d^2\,r}{d\,t^2}+\left(\frac{e\,B}{m}\right)^2 r=\int\frac{r}{d}\,\frac{e\,\varrho}{m\,\varepsilon_0}\,d\,s-\frac{U}{d}. \tag{14}$$

The right member is independent of the space coordinate and a function of time only. Since the coefficients in the left member are also independent of the space coordinate, so is r. Consequently r can be put outside the integral sign. The integral that remains is then the mean value of $e\,\varrho/m\,\varepsilon_0$ between the plates. We wish to stress that also those regions that have no space charge must be included when we calculate the mean value. If the mean is indicated by a bar we find

$$\frac{d^2\,r}{d\,t^2}+\left\{\left(\frac{e\,B}{m}\right)^2-\frac{e\,\bar{\varrho}}{m\,\varepsilon_0}\right\}r=-\frac{U}{d}. \tag{15}$$

Let us first assume that the space charge is concentrated to a finite interval in the y direction and that the plates are at infinite distance from that region. We then find that the charge does not enter our equation at all. The perturbation is the same for all particles in the charged region, which means that the space charge moves as a rigid body. This rigid body can oscillate and the frequency of oscillation is the gyro frequency of the charged particles constituting the space charge. We have thus found

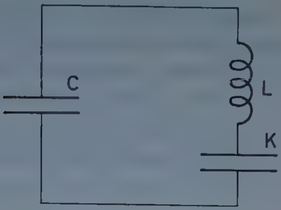

Fig. 2. The admittance of uncompensated electron space charge in a magnetic field for the plane case.

$$C=\frac{1}{\varepsilon_0 d};\quad K=\frac{1}{\varepsilon_0 d}\,\frac{e\bar{\varrho}}{m\,\varepsilon_0}\left\{\left(\frac{e\,B}{m}\right)^2-\frac{e\bar{\varrho}}{m\,\varepsilon_0}\right\}^{-1};\quad L=\varepsilon_0\,d\left(\frac{e\bar{\varrho}}{m\,\varepsilon_0}\right)^{-1}.$$

that the system has an equilibrium position, that this equilibrium is stable for perturbations of the kind we have discussed, and that this is the case whatever the charge distribution within the charged region.

Now if the distance between the plates is finite, the space charge does influence the phenomenon, though only through the mean value of the charge between the plates. This means that, for a given value of the distance between the plates, our phenomenon does depend on the total charge between the plates but is independent of the distribution of the charge. Also in this case, the perturbation has the same value for all particles within the charged region, which means that the charged region moves as a rigid body. Also here we have a natural frequency of oscillation. Its value is smaller than in our previous case by an amount given by the mean density of the space charge.

When the total charge between the plates exceeds a certain critical value, the solution of the homogeneous equation is exponential with real values of the exponent. One of these value is positive. Consequently our equilibrium is unstable. Also in this case, we can find oscillatory solutions of the inhomogeneous equation. Since the equilibrium is unstable the equilibrium cannot be realized in an experiment and thus the solution of the inhomogeneous equation when the total charge between the plates exceeds the critical value is of theoretical interest only.

The admittance for the case when only electrons are present. The admittance is

$$Y=i\,\frac{\omega\,\varepsilon_0}{d}+i\,\frac{\omega\,\varepsilon_0}{d}\,\frac{e\bar{\varrho}}{m\,\varepsilon_0}\left\{\left(\frac{e\,B}{m}\right)^2-\frac{e\bar{\varrho}}{m\,\varepsilon_0}-\omega^2\right\}^{-1}. \tag{16}$$

If we represent the admittance by passive elements, the result will be that given in Fig. 2. C is the capacitance when no charges are present. The space charge acts as a series resonance circuit where the capacitance is positive for a space charge less than the critical value but has a negative value when the space charge exceeds the critical value. Because of the dependence of the corresponding admittance on the frequency it must, despite the sign, be regarded as a capacitance. Here we must also remember that the case of negative capacitance cannot be realized in an experiment because the plasma is not stable in that case.

(c) *Introduction of a stationary positive background*

Also in this case we would like to have r independent of the y coordinate. We are again free to choose any distribution of the electron space charge density. The positive space charge, on the other hand, must be homogeneous in the whole region where electrons are present. This different influence of the positive and the negative space charge density is seen clearly in equation (12), where the negative space charge appears only under the integral in the right member. This means that only the mean value of the negative space charge over the whole region between the plates is of interest. On the other hand, the local positive space charge density enters in the left member. Therefore the negative space charge moves as a rigid body only if the positive space charge is homogeneous.

With the assumption we have made, equation (12) has the form

$$\frac{d^2 r}{d t^2}+\left\{\left(\frac{e B}{m}\right)^2-\frac{e \varrho_+}{m \varepsilon_0}-\frac{e \bar{\varrho}_-}{m \varepsilon_0}\right\} r=-\frac{U}{d}. \tag{17}$$

Here e and m refer to the electron, $\bar{\varrho}_-$ is the mean value of space charge density of the electrons and ϱ_+ is the local value of the space charge density of the psoitive ions. In the third term e is the charge of an electron and ϱ_+ the charge density of the positive ions. Since these two quantities have opposite sign, the third term in equation (17) is positive. This term corresponds to, and is generally known as, the plasma frequency. We thus see that the plasma frequency is given by the charge of the moving particle and the density of the background space charge. The density of the moving species does not enter in this connection. The last term in the left member refers to the mean value of the electron space charge. It has opposite sign to the second and third terms.

If the charges are confined to a finite interval in the y direction but the plates are at infinite distance, the negative space charge density does not enter into equation (17). Our system then has a natural frequency which is the root of the sum of the square of the gyro frequency of the electrons and the so-called plasma frequency defined in the way just discussed. In this case our system is always stable.

If the distance between the plates is finite, the mean value of the electron space charge enters, decreasing the natural frequency of the system. When it exceeds a critical value, the equilibrium of our system is unstable as in our previous example.

In the special case when we have neutrality at every point between the two plates, we can have equilibrium also in the case when the magnetic field vanishes. In this case the natural angular frequency of our system is

$$\left(\frac{e}{m}\,\frac{\varrho_- - \bar{\varrho}_-}{\varepsilon_0}\right)^{\frac{1}{2}}. \tag{18}$$

If the space charge fills only a very small part of the region between the plates, it is equal to the so-called plasma frequency but if the space charge fills the whole region, it is zero.

(d) *Same case with inhomogeneous background charge density*

In the last section we had two species of particles, mobile electrons and stationary ions. We found no difficulty whatever. The space charge distribution of the electrons was a function of the y coordinate, but we had to assume that the background charge was homogeneous. In this section we shall drop this limitation and allow any distribution of the stationary background charge density with the y coordinate.

In the equation of motion for the electrons (12), we find that the right member is independent of the space coordinate. Let us denote it by S. If we then restrict our discussion to the case where r and U are proportional to $e^{i\omega t}$ we find

$$r = S\left\{\left(\frac{eB}{m}\right)^2 - \frac{e\varrho_+}{m\varepsilon_0} - \omega^2\right\}^{-1}, \tag{19}$$

where e and m refer to the electron but ϱ_+ is the positive background space charge density. The latter quantity is a function of the space coordinate and hence also r. If we now introduce our expression for r in the right member of equation (12) and remember that S does not depend on the space coordinate, this right member becomes

$$S\sigma - \frac{U}{d} \quad \text{with} \quad \sigma = \frac{1}{d}\int \frac{e\varrho_-(s)}{m\varepsilon_0}\left\{\left(\frac{eB}{m}\right)^2 - \frac{e\varrho_+(s)}{m\varepsilon_0} - \omega^2\right\}^{-1} ds. \tag{20}$$

But the right member, i.e. this expression, is equal to S. Thus we find

$$S = -\frac{U}{d}\,(1-\sigma)^{-1}. \tag{21}$$

Solutions of the homogeneous equation. In order to get a solution when U vanishes, we must require that the denominator of expression (21) vanishes. Let us assume that ω^2 is an arbitrary complex quantity

$$\omega^2 = p + iq. \tag{22}$$

Here p and q are arbitrary real quantities. The imaginary part of the denominator of (21) is then

$$-i\frac{q}{d}\int \frac{e\varrho_-}{m\varepsilon_0}\left[\left\{\left(\frac{eB}{m}\right)^2 - \frac{e\varrho_+}{m\varepsilon_0} - p\right\}^2 + q^2\right]^{-1} ds. \tag{23}$$

The integrand retains one and the same sign throughout the interval of integration. Expression (23) therefore can vanish only if $q = 0$. Hence the denominator of expression (21) can vanish only if ω^2 is a real quantity. This means that values of ω permitting solutions of the homogeneous equation must be either real or purely imaginary.

The case of a parabolic distribution of the background space charge density. As an example we may study the case when both the positive and negative space charge densities have a parabolic distribution in the y coordinate. We assume the density to be zero on the boundary of the space charge region.

$$\left.\begin{aligned} \frac{e\varrho_-}{m\varepsilon_0} &= a^2\left(1 - \frac{y^2}{\alpha^2}\right) \quad \text{when} \quad -\alpha < y < \alpha \quad \text{and zero elsewhere,} \\ -\frac{e\varrho_+}{m\varepsilon_0} &= b^2\left(1 - \frac{y^2}{\alpha^2}\right) \quad \text{when} \quad -\alpha < y < \alpha \quad \text{and zero elsewhere.} \end{aligned}\right\} \tag{24}$$

We then get three cases

$$(1)\quad \left(\frac{e\,B}{m}\right)^2-\omega^2>0,$$

$$(2)\quad \left(\frac{e\,B}{m}\right)^2<\omega^2<\left(\frac{e\,B}{m}\right)^2+b^2,$$

$$(3)\quad \left(\frac{e\,B}{m}\right)^2+b^2<\omega^2.$$

In the first case it is convenient to introduce σ defined by

$$\sinh\sigma=b\left\{\left(\frac{e\,B}{m}\right)^2-\omega^2\right\}^{-\frac{1}{2}}. \tag{25}$$

We then easily find

$$r=S\left\{\left(\frac{e\,B}{m}\right)^2+b^2-\omega^2-\left(\frac{b\,y}{\alpha}\right)^2\right\}^{-1}, \tag{26}$$

$$S=-\frac{U}{d}\left\{1-\frac{2\,\alpha\,a^2}{d\;\;b^2}+\frac{2\,\alpha\,a^2}{d\;\;b^2}\,\frac{2\,\sigma}{\sinh 2\,\sigma}\right\}^{-1}. \tag{27}$$

In the third case it is convenient to introduce

$$\sin\varphi=b\left\{\omega^2-\left(\frac{e\,B}{m}\right)^2\right\}^{-\frac{1}{2}}. \tag{28}$$

The solution is then

$$r=-S\left\{\omega^2-\left(\frac{e\,B}{m}\right)^2-b^2+\left(\frac{b\,y}{\alpha}\right)^2\right\}^{-1}, \tag{29}$$

$$S=-\frac{U}{d}\left\{1-\frac{2\,\alpha\,a^2}{d\;\;b^2}+\frac{2\,\alpha\,a^2}{d\;\;b^2}\,\frac{2\,\varphi}{\sin 2\,\varphi}\right\}^{-1}. \tag{30}$$

In the second case our expression for η becomes infinite for one value of y. Let us assume that the plasma density and the space derivative are continuous functions of the space coordinate in a region which includes this point. Then, if $y=y_0$ is the singular point

$$\eta=\frac{a}{y-y_0}+f(y). \tag{31}$$

If $\partial\varrho_+/\partial y\neq 0$, a can be so chosen that $f(y_0)$ has a finite value. The electric field shows the same variation with the coordinate. The relative variation of the electric field E across the distance y then becomes

$$\frac{\Delta E}{E}=-\frac{\Delta y}{y-y_0}. \tag{32}$$

If our medium has particle structure we must require that the relative variation of the electric field from one particle to those nearby must be very small in order that we shall be justified in smearing out the particle density to a continuous distribution of charge. Now if Δy in expression (32) is the mean distance between the charged particles of a certain kind, this expression shows that the approximation of a homogeneous medium cannot be used in a certain region round the singular point. How large a region we have to exclude depends on what accuracy we want. Since at present our main interest is to get a qualitatively correct picture, we shall try to simplify the treatment as much as possible. We then put the limit at the point where

$$\left|\frac{\partial E}{\partial y}\,\delta y/E\right| = 1,$$

putting δy equal to the mean inter-electron distance and assume (31) to be valid up to this point.

In an interval in the y direction, which under the present assumption is of the size of two times the mean distance between the particles, we cannot use the concept of a homogeneous region. Since very few particles exist in this region, we can assume that we have vacuum there. The amplitude of the few particles that do exist in this region is limited by non-linear action, so that their influence may be neglected. The dielectric constant on the two boundaries of the vacuum region now has the same absolute value but opposite sign, which means that the electric field retains its magnitude but changes sign across the vacuum gap. In the vacuum gap we have a homogeneous field. In that field those few charged particles present have to be treated as singularities. In the present treatment we shall neglect their presence.

According to the picture we have got we must, when calculating S, put $\varrho(y) = 0$ when $y_0 - \delta < y < y_0 + \delta$. We cannot expect any greater accuracy from such a procedure, since at the boundary of the region where we have assumed a continuous distribution of charges this approximation does not hold. Nevertheless we may expect a qualitatively valid result.

We introduce the quantity ϑ defined by

$$\cosh \vartheta = b\left\{\omega^2 - \left(\frac{e\,B}{m}\right)^2\right\}^{-\frac{1}{2}}. \tag{33}$$

If we let δ tend to zero we obtain the solution in the form

$$\left.\begin{aligned} r &= S\left\{\left(\frac{e\,B}{m}\right)^2 + b^2 - \omega^2 - \left(\frac{b\,y}{\alpha}\right)^2\right\}^{-1}, \\ S &= -\frac{U}{d}\left\{1 - \frac{2\,\alpha}{d}\frac{a^2}{b^2} - \frac{2\,\alpha}{d}\frac{a^2}{b^2}\frac{2\,\vartheta}{\sinh 2\,\vartheta}\right\}^{-1}. \end{aligned}\right\} \tag{34}$$

We find the value of ω^2 for which the homogeneous equation (12) has a solution when we put the denominator of S equal to zero. An examination shows that there exists no solution for any value of ω^2 when $0 < (2\alpha/d)/(a^2/b^2) < \frac{1}{2}$. This means that if the absolute value of the ratio of the electron space charge density and the background space charge density is below a certain value we find no resonance.

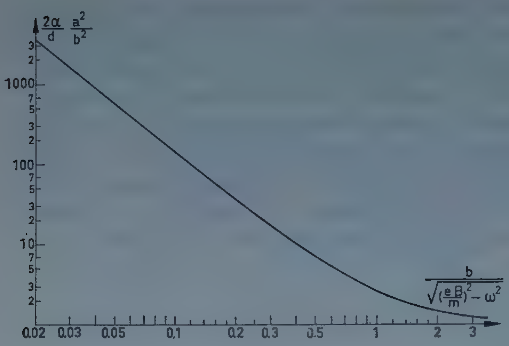

Fig. 3

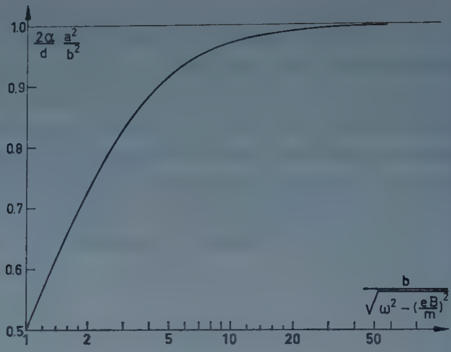

Fig. 4

Fig. 3. Resonance condition when $\omega^2 < (eB/m)^2$ for a parabolic spatial charge distribution with stationary background space charge.

Fig. 4. Resonance condition when $(eB/m)^2 < \omega^2 < (eB/m)^2 + b^2$ for a parabolic spatial charge distribution with stationary background space charge.

When $(2\alpha/d)\,(a^2/b^2) > 1$ we have resonance for a frequency satisfying $\omega^2 < (eB/m)^2$. Fig. 3 shows that the quantity $(2\alpha/d)\,(a^2/b^2)$ giving resonance tends towards 1, when the applied frequency tends to the gyro frequency of the electrons. When we increase $(2\alpha/d)\,(a^2/b^2)$ the value of ω^2 giving resonance decreases, passes zero and decreases to $-\infty$. When the electron space charge density exceeds the background space charge density by a certain amount, our configuration is thus unstable. The limiting value depends on the ratio $b/(eB/m)$. When this ratio is near zero the limit corresponds to $(4/3)/(\alpha/d)a^2 \approx (eB/m)^2$, which means that the electron density for the limit of the stable region is given by the gyro frequency of the electrons and not by the background charge density.

Fig. 4 refers to the case when there is a singularity somewhere in the plasma region. Then the resonance frequency satisfies the relation $(eB/m)^2 < \omega^2 < (eB/m)^2 + b^2$. Further $\frac{1}{2} < (2\alpha/d)\,(a^2/b^2) < 1$.

Dissipative medium. If we allow collisions between the particles, the amplitude is limited everywhere in space. Generally the collisions are taken into account by the introduction of a friction term in the equation of motion. If the losses are sufficiently large we can use the concept of a homogeneous medium throughout space. Let us now assume a frequency corresponding to case 2 as introduced in the beginning of paragraph 2.1*d*. If we then let the losses tend continuously towards zero we find—in the limit—another result than the one we have just presented. If, however, we remember that the simpler way of solving the problem may be used only as long as the relative variation of the electric field is below a certain value throughout the plasma region, we find that, when we decrease the losses of our medium, we eventually arrive at this limiting value. Following these lines of thought we obtain the solution corresponding to the one found in literature, when the losses are sufficiently large, and the one we have presented in this paper, when the losses tend to zero. We thus find the same solution if we first introduce losses and let them tend to zero as if we start with the assumption of no losses.

(e) *A neutral and homogeneous medium with several species of particles*

If we restrict the discussion to a homogeneous medium containing several species of particles we will find that all r are independent of the space coordinate. This means that they can be put outside the integral in the right member of equation (12). If we also assume that the medium is neutral and confined to a layer of thickness 2α, equation (12) becomes

$$\left.\begin{aligned} \left\{\left(\frac{e_\mu B}{m_\mu}\right)^2-\omega^2\right\} r_\mu+\sum_\nu a_\nu^2 r_\nu &= -\frac{U}{d}, \\ a_\mu^2 &= \left(1-\frac{2\alpha}{d}\right)\frac{e_\mu \varrho_\mu}{m_\mu \varepsilon_0}. \end{aligned}\right\} \tag{35}$$

The sum

$$\sum_\nu a_\nu{}^2 r_\nu$$

is the same in all equations. If we transfer the sum to the right member, this member will thus be identical for all equations. Let us denote it by S. The solution is then

$$r_\mu = S\left\{\left(\frac{e_\mu B}{m_\mu}\right)^2-\omega^2\right\}^{-1}, \tag{36}$$

Introducing these values in the equation defining S we find

$$S=-\frac{U}{d}\left\{1+\sum_\nu \frac{a_\nu^2}{\left(\frac{e_\nu B}{m_\nu}\right)^2-\omega^2}\right\}^{-1}. \tag{37}$$

In Fig. 5 the denominator of S is given as a function of ω^2. We there have an asymptote at the gyro frequency of each species. We can also see that the number of zeros is equal to the number of species. But this is equal to the number of roots of the equation. All roots in ω^2 are positive and hence we can conclude that all values of ω that make $S^{-1}=0$ are real in the case of a homogeneous neutral medium with any number of species of charged particles and a homogeneous magnetic field. At these zeros, S

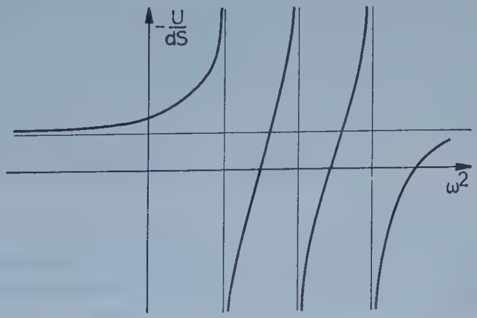

Fig. 5. S–$1(\omega^2)$ for a neutral and homogeneous medium with three species of particles.

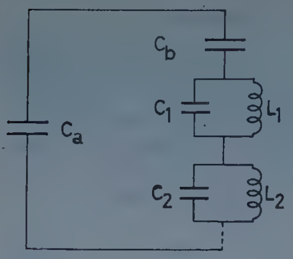

Fig. 6. The equivalent circuit for a neutral and homogeneous medium.

becomes infinite and consequently all the r_μ become infinite. They therefore represent resonances of our system. When $U=0$ we can have a finite amplitude of oscillation at these frequencies. These oscillations thus represent natural modes of our system.

At the gyro frequency of the μth species we easily find

$$\left.\begin{aligned} r_\mu &= -\frac{U}{d}\,a_\mu^{-2}, \\ r_\nu &= 0 \quad \text{when} \quad \nu \neq \mu. \end{aligned}\right\} \quad \text{at} \quad \omega = \pm\frac{e_\mu B}{m_\mu}. \tag{38}$$

We thus see that when ω is equal to the gyro frequency of the μth species the amplitude of motion for this species is finite and non-zero but zero for all others.

Since we have found that all the natural modes of our system are oscillatory we must conclude that the equilibrium of our system is stable for perturbations of the kind we have discussed. In the special case when we let the magnetic field strength tend to zero we find

$$r_\mu = -\frac{U}{d}\left(\sum_\nu a_\nu^2 - \omega^2\right). \tag{39}$$

We find that we have only one resonance frequency and that that is a real quantity. The natural mode of oscillation is thus also in this case oscillatory and our system is stable. The resonance frequency is the root of the sum of the weighted squares of the so-called plasma frequencies for the different kinds of particle present. The weight depends on the geometry.

Our discussion shows that the introduction of a magnetic field splits the single resonance which exists when the magnetic field is zero into as many resonances as there are kinds of particles present. All these frequencies, except the highest one, approach zero if we decrease the magnetic field to zero.

The admittance. The admittance is

$$Y = i\frac{\omega\,\varepsilon_0}{d}\left(1 + \frac{2\,\alpha/d}{1-2\,\alpha/d}\,\frac{\sigma}{1-\sigma}\right) \quad \text{with} \quad \sigma = \sum_\nu a_\nu^2 \left\{\left(\frac{e_\nu B}{m_\nu}\right)^2 - \omega^2\right\}^{-1}. \tag{40}$$

Some simple calculations give us the circuit shown in Fig. 6. The contribution of each species is represented as a parallel resonance circuit with the resonance frequency equal to its gyro frequency. All circuits are coupled in series and in series with them all we have the capacity C_b which, however, is negative. Because of the variation with frequency of its impedance we must accept C_b as a capacitance. In parallel with this branch we have a capacitance C_a whose value is equal to the vacuum capacitance between the plates.

(f) *Non-neutral homogeneous medium*

Also in a medium that is not neutral we can proceed very similarly. Let us call the excess charge density Q, i.e. $\sum_\nu \varrho_\nu = Q$. We then find

$$\left.\begin{aligned} \frac{e_\mu Q}{m_\mu \varepsilon_0} &= \sum_\nu \frac{e_\mu \varrho_\nu}{m_\mu \varepsilon_0}, \\ \left\{\left(\frac{e_\mu B}{m_\mu}\right)^2 - \frac{e_\mu Q}{m_\mu \varepsilon_0} - \omega^2\right\} r_\mu &= S, \\ S &= -\sum_\nu a_\nu^2 r_\nu - \frac{U}{d}. \end{aligned}\right\} \tag{41}$$

From these expressions we find, in the same way as in the previous example,

$$\left.\begin{aligned} r_\mu &= S\left\{\left(\frac{e_\mu B}{m_\mu}\right)^2 - \frac{e_\mu Q}{m_\mu \varepsilon_0} - \omega^2\right\}^{-1}, \\ S &= -\frac{U}{d}\left\{1 + \sum_\nu \frac{a_\nu^2}{\left(\frac{e_\nu B}{m_\nu}\right)^2 - \frac{e_\nu Q}{m_\nu \varepsilon_0} - \omega^2}\right\}^{-1}. \end{aligned}\right\} \tag{42}$$

If we introduce $\{(e_\mu B/m_\mu)^2 - e_\mu Q/m_\mu \varepsilon_0\}^{\frac{1}{2}}$ instead of the gyro frequency in the previous case we can even now use the figure corresponding to the neutral case. We then find that the asymptotes for those species that have a charge opposite to the excess charge always exist. The corresponding frequencies increase when we increase the absolute value of the excess charge. The frequency of the asymptotes for a species having the same sign of charge as the excess charge, decreases with increasing absolute value of the excess charge. This means that, for a certain value of the excess charge, the asymptote reaches the frequency zero. For a larger absolute value of the excess charge the corresponding asymptote does not exist any more. Each time this happens when we increase Q, the equation $S^{-1} = 0$ loses two real roots and two imaginary roots appear.

The disappearance of the asymptote occurs first for that particle with charge of appropriate sign which has the smallest value of e/m.

Hence we have found that our system is stable if the excess charge does not exceed a certain limit (but that it is unstable if the excess charge exceeds this limit). The limit is different for different sign of the excess charge. We also find that the range of stability cannot be increased by introducing new kinds of particles or by changing the densities of those already present assuming constant excess charge. Of course, we can decrease the range by introducing new kinds of particles.

(g) *The general case*

We shall study the general case of an inhomogeneous plasma consisting of several species of particles. We also allow it to be non-neutral. The last term and the two sums are the same for all the equations (12). Let us call the sum of all these terms S. Since the first of the component sums is a function of the space coordinate, so is S. If we restrict the discussion to the case where r varies harmonically with time, i.e. r is proportional to $e^{i\omega t}$, we find

$$r_\mu = S\left\{\left(\frac{e_\mu B}{m_\mu}\right)^2 - \omega^2 - \frac{e_\mu Q}{m_\mu \varepsilon_0}\right\}^{-1}, \tag{43}$$

$$S = -\sum_\nu \frac{e_\nu \varrho_\nu}{m_\nu \varepsilon_0} r_\nu + \sum_\nu \int \frac{r_\nu}{d} \frac{e_\nu \varrho_\nu}{m_\nu \varepsilon_0} d s - \frac{U}{d}. \tag{44}$$

In equation (44) the two last terms are independent of y. Let us call their sum k. We then find

$$S = k\,\{1+\sigma\}^{-1}. \tag{45}$$

$$k = \frac{1}{d}\int S\sigma\, d s - \frac{U}{d} \quad \text{with} \quad \sigma = \sum_\nu \frac{e_\nu \varrho_\nu}{m_\nu \varepsilon_0}\left\{\left(\frac{e_\nu B}{m_\nu}\right)^2 - \frac{e_\nu Q}{m_\nu \varepsilon_0} - \omega^2\right\}. \tag{46}$$

If we introduce the expression (45) for S in (46) remembering that k is independent of the space coordinate we get the following expression for k

$$k = -\frac{U}{d}\left\{1 - \frac{1}{d}\int \frac{\sigma}{1+\sigma} d s\right\}^{-1} = -\frac{U}{d}\left\{1 - \frac{2\alpha}{d} + \frac{1}{d}\int \frac{d s}{1+\sigma}\right\}^{-1}. \tag{47}$$

Hence our solution can be written as

$$r_\mu = \left\{\left(\frac{e_\mu B}{m_\mu}\right)^2 - \frac{e_\mu Q}{m_\mu \varepsilon_0} - \omega^2\right\}^{-1} (1+\sigma)^{-1} \left(1 - \frac{2\alpha}{d} + \frac{1}{d}\int \frac{d s}{1+\sigma}\right)^{-1} \left(-\frac{U}{d}\right). \tag{48}$$

Thus r is expressed as a product of three factors. The first one is specific, whereas the second is the same for all species and gives the common space dependence. The third factor is also the same for all particles. It is independent of the space coordinate.

When the first factor becomes infinite for some species, the second vanishes in such a way that the product is finite and non-zero. This means that the amplitude is finite and non-zero for this species but zero for all others. For a neutral plasma this happens at the gyro frequency and applies to the whole plasma region. When the excess charge density is independent of the space coordinate this phenomenon concerns the whole plasma region but appears at frequencies which are modified by the excess charge density. When the excess space charge density is a function of the space coordinate, this phenomenon occurs only at discrete value of the space coordinate.

We shall discuss the third factor for the special case of a neutral plasma, assuming the following space dependence

$$\frac{e_\mu \varrho_\mu}{m_\mu \varepsilon_0} = b_\mu^2\left(1 - \frac{y^2}{\alpha^2}\right). \tag{49}$$

We then find that the denominator, D, of the third factor can be written

$$D = 1 - \frac{2\alpha}{d} + \frac{2\alpha}{d}\int_0^1 \frac{d s}{1+u-u s^2} \quad \text{with} \quad u = \sum \frac{b_\nu^2}{\left(\frac{e_\nu B}{m_\nu}\right)^2 - \omega^2}. \tag{50}$$

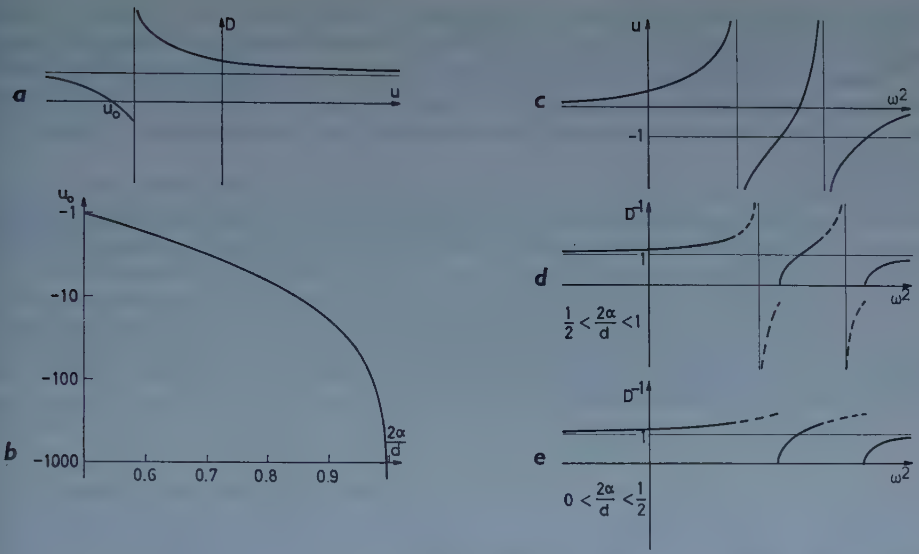

Fig. 7. The frequency characteristic for a plasma region with two particle species and parabolic spatial distribution.

(*a*) D of equations (51), (52) and (53).
(*b*) The zero of D as a function of $2\alpha/d$.
(*c*) The parameter u of equation (50) as a function of frequency ω.
(*d*) D as a function of ω in the case when there are resonances. In the dashed portion there is a singular layer in the plasma.
(*e*) D as a function of ω in the case when there are no resonances. In the dashed portion there is a singular layer in the plasma.

As in the earlier discussion, we have three cases

(*1*) $u>0$. We introduce sh $\sigma=(u)^{\frac{1}{2}}$ and find

$$D=1-\frac{2\alpha}{d}+\frac{2\alpha}{d}\,\frac{2\sigma}{\operatorname{sh}2\sigma}, \tag{51}$$

(2) $-1<u<0$. We introduce $\sin\varphi=(-u)^{\frac{1}{2}}$ and find

$$D=1-\frac{2\alpha}{d}+\frac{2\alpha}{d}\,\frac{2\varphi}{\sin 2\varphi}, \tag{52}$$

(*3*) $u<-1$. We introduce $\cosh\vartheta=(-u)^{\frac{1}{2}}$. If we now, as before, exclude a neighbourhood of the point where the integral becomes infinite, we find after integration

$$D=1-\frac{2\alpha}{d}-\frac{2\alpha}{d}\,\frac{2\vartheta}{\sinh 2\vartheta}. \tag{53}$$

In Fig. 7 *a* we have given D as a function of u. We find that D has one zero when $2\alpha/d>\frac{1}{2}$ but none when $2\alpha/d<\frac{1}{2}$. In Fig. 7*b* the zero u_0 is given as a function of $2\alpha/d$.

For the case of two species, Fig. 7*c* gives u as a function of ω^2. The asymptotes are found at the gyro frequencies. Using Figs. 7*a* and 7*c* we can derive D^{-1} as a function of ω^2. Fig. 7*d* illustrates the case when there are resonances, and we find that there are as many resonances as there are kinds of particles. Fig. 7*e* illustrates the case when there are no resonances. The resonances have been replaced by peaks of finite amplitude.

In the last two figures there are points of discontinuity. They occur at the frequencies that give infinite amplitude in the densest part of the plasma region. If we decrease the frequency from a point of discontinuity, the region where we have infinite amplitude moves towards parts where the plasma density is smaller and, at the gyro frequency, reaches the edge of the plasma region. Consequently we find resonance of a maximum value of D^{-1} in the frequency range giving infinite amplitude somewhere in the plasma region. Resonance cannot occur when the amplitude is finite throughout the plasma region.

(h) *Discontinuous density distribution*

It is generally quite complicated to discuss the electromagnetic behaviour of an inhomogeneous plasma region. Therefore one is inclined to approximate the actual density distribution by a number of steps, each with constant density. In order to investigate the consequences of such a procedure we shall here discuss some calculations of this kind. For our purpose we may treat the case where there are only two species, the same in all regions, assuming neutrality in each region separately and also that there is no magnetic field. The number of homogeneous regions is arbitrary.

If we—in equation (12)—exclude the term related to the magnetic field and the one related to the excess charge density we find that the second derivative of the position vector has the same value for all species. If we assume harmonic variation in time, also r must have the same value for all kinds of particles at the same place. This also means that we can use one and the same equation for them all. In our case the number of equations is reduced by half. We get one equation for each homogeneous region. The right member of equation (12) has the same value in all these equations. Let us denote it by S. Letting the index of r refer to the region where the particle is situated, we obtain from equation (12)

$$\ddot{r}_\mu + \Omega_\mu^2 r_\mu = S, \tag{54}$$

$$S = \sum_\nu \frac{2\,\alpha_\nu}{d}\,\Omega_\nu^2 r_\nu - \frac{U}{d}, \tag{55}$$

where

$$\frac{e_I \varrho_{I\mu}}{m_I \varepsilon_0} + \frac{e_{II} \varrho_{II\mu}}{m_{II} \varepsilon_0} = \Omega_\mu^2. \tag{56}$$

Here e_I and m_I refer to one and e_{II} and m_{II} to the other. If the phenomenon is harmonic with the angular frequency ω equation (54) gives

$$r_\mu = S\,(\Omega_\mu^2 - \omega^2)^{-1}. \tag{57}$$

Introducing this equation in (55) we find

$$S = -\frac{U}{d}\left\{1 - \sum_\nu \frac{2\,\alpha_\nu}{d}\,\Omega_\nu^2/(\Omega_\nu^2 - \omega^2)\right\}. \tag{58}$$

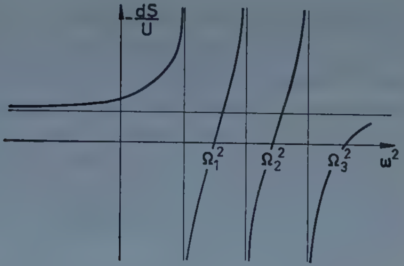

Fig. 8. The frequency characteristic for a plasma consisting of three homogeneous regions.

In Fig. 8 the value of S is given for the case of three homogeneous regions. We easily find that there are as many asymptotes as homogeneous regions or equivalently as many resonances as regions. We also see that S vanishes at each of the frequencies defined by (56) but the denominator of (57) becomes zero for one of the regions. While the particles in the μth region have a finite amplitude at the frequency Ω_μ, the amplitude vanishes in the other regions.

Thus, if we try to approximate a region where the space derivative of the charge density is continuous and non-zero everywhere in this way the number of resonances equals the number of steps introduced. Since the previous discussion has shown that such a system has only one resonance we must reject this "approximation".

2.2 *Introduction to the case of finite temperature*

(a) *Only one velocity represented in the unperturbed case. Neutralizing stationary background*

A non-zero temperature is generally introduced by means of a pressure term in the macroscopic equation of motion. In the present paper we shall instead generalize the results of the previous chapter by introducing an unperturbed motion of those particles constituting the plasma. The development is not carried beyond the initial stage where the velocity spectrum consists of one discrete line only.

We have shown that the motion of a charged region of one species is independent of the charge density distribution if the distance between the two plates is infinite but the charged region has a finite width. When there are two particle species with different values of e/m the situation is different. Now the constitution of the plasma is of interest also in the case of infinite distance between the plates. The important difference between these two cases is that in the first the particle velocity is a single-valued function of the space coordinate but in the latter the particle velocity has two values at every point in space, one for each species. The important factor is now the differential velocity.

In the unperturbed state we have only one kind of particle and that moves back and forth periodically in the direction in which a perturbing electric field is applied. Also now there are two values of the velocity represented at each point in space, which is true also if we disregard the unperturbed part of the velocity vector and look only at the perturbation in the velocity. This means that the discussion of a charged region with finite temperature belongs to the second group of phenomena of the previous discussion.

As before, we shall introduce only one new feature at a time. We first assume that we have neutrality when there is no perturbing field. The neutralizing field may be due to a stationary space charge. Then, the unperturbed motion is linear in time inside the charged region. We shall simply assume that the particles are reflected at $y=0$ and $y=d$. We also assume that the unperturbed absolute value of the velocity is the same for all particles, and an equal probability for all positions between $y=0$ and $y=d$. Finally, we assume that the perturbing electric field in the y-direction is obtained from plates at such great distance from each other that the distance can be treated as infinite.

Integration of the equation of motion for an individual particle would be very much simplified if we need not consider the reflection of the particles. This can be achieved if we introduce a mathematical space which coincides with the physical one when $0 < y < d$. The mathematical space from 0 to $-d$ is obtained by reflection of the physical space in the surface $y=0$. By repeated reflections in the surfaces $y=0$ and $y=d$ we can correlate corresponding points in the physical and the mathematical space.

In the region 0 to d the electric field has the same value and direction in the mathematical and the physical space. In the region $-d$ to 0 in the mathematical space motion in the positive direction corresponds to motion in the negative direction in the physical space. This means that the electric field in this part must have a direction opposite to that in the physical space. Thus the electric field in the mathematical space must change sign for each distance d passed.

We have just found that the electric field must be an odd function of the y-coordinate and at the same time periodic in space with the period $2d$. Assuming harmonic time variation, the most general field satisfying these conditions is

$$F=\sum_{n=1}^{\infty} A_n \sin n\pi\frac{y}{d}\, e^{i\omega t}, \tag{1}$$

where we have used the symbol F for the field, since we want to use E for the part of the field due to the potential difference between the plates.

The equation of motion in the mathematical space is now

$$m\ddot{y}=eF, \tag{2}$$

where F is the value of the electric field at the perturbed position of the charged particle. In equation (2) m is the mass of the particle and e its charge.

In an experiment we always have a continuous velocity distribution. To simplify the treatment we shall, however, assume that all the particles have the same absolute value of the velocity in the unperturbed state. Half the number of particles move upwards and half downwards. The density in the physical space may be N.

In the mathematical space all particles move upwards. All particles have thus, in the unperturbed case, the same direction and absolute value of the velocity. The density is $\frac{1}{2}N$. If the particles move along the y axis, the position y_0 of a certain particle in the unperturbed case is given by

$$y_0=vt+a, \tag{3}$$

where v is the velocity of the particle, t the time and a a phase constant. This phase constant is necessary because at any instant all positions in space are equally probable. In the perturbed case the particles are situated a distance η out of the unperturbed positions. If y is the perturbed position we have

$$y = y_0 + \eta\,(y_0, t). \tag{4}$$

If we now introduce (4) in (2) we find

$$\frac{d^2\eta}{dt^2} = \sum_{n=1}^{\infty} \frac{e A_n}{m} \sin n\pi \frac{y_0+\eta}{d}\, e^{i\omega t}. \tag{5}$$

We assume $n\pi\eta/d \ll \frac{1}{2}\pi$. We would like to approximate the cosine of this quantity by 1 and the sine by the argument itselt. This is possible only if we restrict n to a finite region. In this section we shall formally introduce an unlimited series in n as we know that in an experiment a very limited number of terms are of interest. Looking at the treatment from this point of view our procedure can be justified, even if it is not fully correct from a formal point of view. With this approximation equation (5) becomes

$$\frac{d^2\eta}{dt^2} - \sum_{n=1}^{\infty} \left(\frac{e A_n}{m}\frac{n\pi}{d} \cos \frac{n\pi y_0}{d}\right) e^{i\omega t}\, \eta = \sum_{n=1}^{\infty} \frac{e A_n}{m} \sin n\pi \frac{y_0}{d} e^{i\omega t}. \tag{6}$$

This is an inhomogeneous linear differential equation with periodic coefficients. If we keep A_n sufficiently small, i.e. we treat only small perturbations, we can, as a first approximation, put the coefficient of η equal to zero and obtain an equation with constant coefficients. If we then introduce (3) into (6) we find

$$\frac{d^2\eta}{dt^2} \approx \sum_{n=1}^{\infty} \frac{e A_n}{2im} \exp\left(i\omega t + i n\pi\frac{vt}{d} + i n\pi\frac{a}{d}\right) - \sum_{n=1}^{\infty} \frac{e A_n}{2im} \exp\left(i\omega t - i n\pi\frac{vt}{d} - i n\pi\frac{a}{d}\right). \tag{7}$$

If we integrate and then introduce y_0 again we find

$$\eta = \sum_{n=1}^{\infty} \frac{e A_n}{2im} \frac{\exp\left(i\omega t - i\frac{n\pi}{d}(y-\eta)\right)}{\left(\omega - n\pi\frac{v}{d}\right)^2} - \sum_{n=1}^{\infty} \frac{e A_n}{2im} \frac{\exp\left(i\omega t + i\frac{n\pi}{d}(y+\eta)\right)}{\left(\omega + n\pi\frac{v}{d}\right)^2}. \tag{8}$$

Here we would again like to neglect the terms $n\pi\eta/d$. The same difficulty as above appears here and we treat it in the same way. We find

$$\eta = \sum_{n=1}^{\infty} \frac{e A_n}{2im} \frac{\exp\left(i\omega t - i n\pi\frac{y}{d}\right)}{\left(\omega - n\pi\frac{v}{d}\right)^2} - \sum_{n=1}^{\infty} \frac{e A_n}{2im} \frac{\exp\left(i\omega t + i n\pi\frac{y}{d}\right)}{\left(\omega + n\pi\frac{v}{d}\right)^2}. \tag{9}$$

There are two sources of the electric field. The first is the potential difference between the plates, which are at infinite distance in the physical space. The other source is the space charge distribution.

The first source creates a homogeneous electric field $E = E_0\, e^{i\omega t}$ in the physical space. In the mathematical space it is a square wave with the period $2d$. Hence it can be written

$$E = \frac{4}{\pi}\, E_0 e^{i\omega t} \sum_{n=0}^{\infty} (2n+1)^{-1} \sin\,(2n+1)\,\frac{\pi y}{d}. \tag{10}$$

When we discuss the contribution to the electric field from the space charge, we have to remember that at any point y in the physical space we have particles moving upwards and particles moving downwards. The position of those particles moving upwards is the same in the physical and the mathematical space. Those moving downwards at the point y in the physical space are situated at $-y$ in the mathematical space. Further the perturbation for those particles moving upwards in the physical space is the same in the physical and the mathematical space but the perturbation for those particles which move downwards in the physical space has different sign in the physical and the mathematical spaces. If we then also remember that the density in the physical space is N but in the mathematical space $\frac{1}{2}N$ we easily find the space charge contribution to the electric field at the position y

$$(\tfrac{1}{2}Ne/\varepsilon_0)\,\eta(-y)-(\tfrac{1}{2}Ne/\varepsilon_0)\,\eta(y). \tag{11}$$

Thus the total electric field is

$$F(y)=E+(\tfrac{1}{2}Ne/\varepsilon_0)\,\eta(-y)-(\tfrac{1}{2}Ne/\varepsilon_0)\,\eta(y). \tag{12}$$

If we introduce the expression (10) for E and (9) for η in the relation (12) we can compare this relation for the electric field with the expression for the field (1) introduced in the beginning of this section. The comparison between equation (1) and (12) gives

$$\frac{4}{\pi}\,\frac{E_0}{2n+1}+\frac{\Omega^2}{2}A_{2n+1}\left[\left\{(2n+1)\frac{\pi v}{d}+\omega\right\}^{-2}+\left\{(2n+1)\frac{\pi v}{d}-\omega\right\}^{-2}\right]=A_{2n+1}, \tag{13}$$

$$\frac{\Omega^2}{2}A_{2n}\left[\left\{2n\frac{\pi v}{d}+\omega\right\}^{-2}+\left\{2n\frac{\pi v}{d}-\omega\right\}^{-2}\right]=A_{2n}, \tag{14}$$

with

$$\frac{Ne^2}{m\varepsilon_0}=\Omega^2. \tag{15}$$

The unknown coefficients A_n can be found very easily as each equation contains only one unknown. The equations for the coefficients with even indices do not contain the applied field E and therefore correspond to modes which cannot be excited by fields of the type discussed. The coefficients with odd indices depend on the applied field and can thus be excited.

If we solve equation (13) for A_{2n+1}, we find

$$A_{2n+1}=\frac{4}{\pi}\,\frac{E_0}{2n+1}\left[1-\tfrac{1}{2}\Omega^2\left\{(2n+1)\frac{\pi v}{d}+\omega\right\}^{-2}-\tfrac{1}{2}\Omega^2\left\{(2n+1)\frac{\pi v}{d}-\omega\right\}^{-2}\right]^{-1}. \tag{16}$$

This is a particular solution for the equation when E is different from zero, provided that none of the denominators vanish. If one of them does vanish we cannot find a stationary solution.

The normal modes. Even in the case where we do not have an applied field E, equations (13) and (14) may have solutions, but only if the following relation is satisfied

$$\tfrac{1}{2}\Omega^2\left[\left(\frac{n\pi v}{d}+\omega\right)^{-2}+\left(\frac{n\pi v}{d}-\omega\right)^{-2}\right]=1. \tag{17}$$

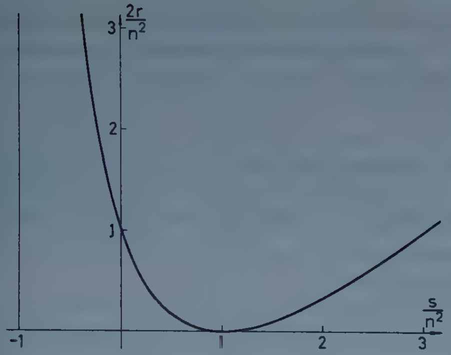

Fig. 9

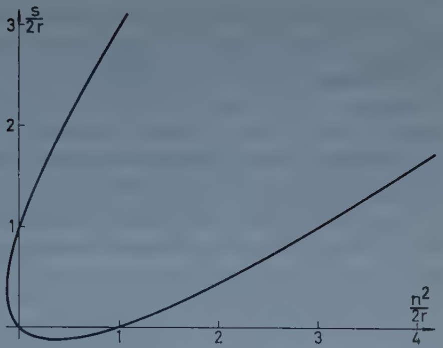

Fig. 10

Fig. 9. Graphical representation of equation (19), especially useful when the frequency of oscillation is given.

Fig. 10. Graphical representation of equation (19), especially useful when the plasma density is given.

Let us introduce the notations

$$r=\tfrac{1}{2}\,\Omega^2\Big/\frac{\pi^2 v^2}{d^2} \text{ and } s=\omega^2\Big/\frac{\pi^2 v^2}{d^2}. \tag{18}$$

Equation (17) can then be rewritten as

$$2\,r=\frac{(s-n^2)^2}{s+n^2}. \tag{19}$$

This is the equation of a hyperbola. Only the branch for which $r>0$ is of interest here. It is reproduced in Fig. 9.

The figure shows that when

$$\Omega>\frac{n\pi v}{d} \tag{20}$$

equation (19) allows solutions for $s<0$, which means that we have natural modes with exponentially increasing and decreasing amplitude. Thus, if relation (20) is satisfied for integral values of n different from zero, the plasma is unstable. The number of unstable modes equals the nearest integral number below $\Omega d/\pi v$. In the present case we have assumed that the plasma is confined by reflecting walls to the volume between the planes $y=0$ and $y=d$, and therefore the instability cannot cause an increase in volume. Thus the instability can only cause a change in the velocity distribution. We find that an one-velocity distribution is unstable if the common velocity in the unperturbed case is such that the time of flight of particles from one wall to the other and back again is larger than the period $2\pi/\Omega$. Our theory is too simple to tell what velocity distribution is necessary to give stability. However, we may state that, if the velocity of the particles is so large that the relation (20) is not satisfied for any integral value of n, the plasma is stable for the kind of perturbations discussed here.

Let us assume that ω, v and d are given and that we want to know the plasma densities corresponding to natural modes. In Fig. 9 we can see that the density is a single-valued function of the parameter n. If we first discuss periodic disturbances we can distinguish two cases. The first is characterized by $\omega > \pi v/d$, i.e. $s > 1$. Fig. 9 shows that we then have a limited number of modes for which Ω decreases with n^2. Apart from these, there is an infinite series of modes for which the plasma density is a monotonically increasing function of n^2.

The second case for periodic disturbances is characterized by $\omega < \pi v/d$, i.e. $s < 1$. Fig. 9 shows that the plasma density corresponding to natural modes is then an infinite series, the terms increasing monotonically with n^2.

We now come to the case where we have exponentially increasing or decreasing perturbations, i.e. $s < 0$. For rapidly growing solutions, i.e. for $s < -n^2$, we have no natural modes for any value of the charge density. For solutions growing slowly enough we have an infinite number of solutions for integers n satisfying $n > (-s)^{\frac{1}{2}}$.

Let us now assume that Ω, v and d are given. Then, for each value of n, equation (19) gives two values of ω, one value corresponding to the branch $s/n^2 < 1$ in Fig. 9 and one value corresponding to the other branch of the curve in the same figure. This is seen more clearly in Fig. 10. We find that both values of ω increase with n as long as we consider only oscillating solutions.

In the domain of the complex plane for ω obtained when excluding the axes, we find no solutions corresponding to natural modes.

The impedance. Let us now assume that we have two plane metal plates at the finite distance D and that the region containing the charges is inside the smaller region d ($d < D$). In the region where no charges are present we then have the field E, which we introduced previously. In the charged region we have the field F. The potential difference between the plates is given by $E \cdot D$ with any specified relative accuracy provided D is sufficiently great. In the present case when D has a finite value the distribution of the charges also contributes to the potential difference.

The potential difference across the plasma region U_p is

$$U_p = -\int_0^d F\,dy. \tag{21}$$

But we have found that (see equations (14) and (16))

$$A_{2n} = 0 \quad \text{and} \quad A_{2n+1} = \frac{4}{\pi}\,\frac{E_0}{2n+1}\,(1+\vartheta) \tag{22}$$

with

$$\vartheta = \Omega^2 \left[\omega^2 + \left\{(2n+1)\frac{\pi v}{d}\right\}^2\right] \left(\left[\omega^2 - \left\{(2n+1)\frac{\pi v}{d}\right\}^2\right]^2 - \Omega^2\left[\omega^2 + \left\{(2n+1)\frac{\pi v}{d}\right\}^2\right]\right)^{-1}. \tag{23}$$

Introducing this expression in the definition of U_p, remembering that F is defined by equation (1), we find that the first term in A_{2n+1} gives the potential difference $-E_0 d\, e^{i\omega t}$, which is due to the applied field E. The other term is due to the space charge.

After performing the integration we find

$$U_p = -E_0 d - E_0 d \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\vartheta}{(2n+1)^2}. \tag{24}$$

If we write the terms as a sum of partial fractions we easily arrive at the Riemann ζ-function and the Ψ functions in performing the summation [23]. Instead of the Ψ-functions we can introduce trigonometric functions. Finally, we arrive at

$$U_p = -E d - \tfrac{1}{2} E d \frac{\Omega^2}{\omega^2 - \Omega^2} (1 + p_1 - p_2), \tag{25}$$

$$\text{where} \qquad x_{1,2} = r + s \pm (r^2 + 4rs)^{\frac{1}{2}} \quad \text{and} \quad p_{1,2} = \tfrac{1}{2} \frac{3s - x_{1,2}}{(r^2 + 4rs)^{\frac{1}{2}}} \frac{\operatorname{tg} \frac{1}{2}\pi x_{1,2}^{\frac{1}{2}}}{\frac{1}{2}\pi x_{1,2}^{\frac{1}{2}}}. \tag{26}$$

Observing that, in the physical space, the field outside the plasma region is E we find, when the distance between the plates is D, the impedance of the system

$$Z = \frac{D}{i\omega\varepsilon_0} + \frac{d}{i\omega\varepsilon_0} \frac{\frac{1}{2}\Omega^2}{\left(\frac{\pi v}{d}\right)^2} \frac{\Omega^2}{\omega^2 - \Omega^2} (1 + p_1 - p_2). \tag{27}$$

Here, the factor $\omega^2 - \Omega^2$ appears in the denominator and it is of interest to determine the value of Z when $\omega^2 = \Omega^2$. We easily find

$$Z = \frac{D}{i\Omega\varepsilon_0} + \frac{d}{i\Omega\varepsilon_0} \frac{2}{9} \left(-2 - \frac{\pi^2}{4} + 2 \frac{\operatorname{tg} \frac{1}{2}\pi (6r)^{\frac{1}{2}}}{\frac{1}{2}\pi (6r)^{\frac{1}{2}}} \right) \quad \text{when} \quad \omega^2 - \Omega^2 = 0. \tag{28}$$

Consequently the impedance has a finite value at $\omega^2 = \Omega^2$ if $(6r)^{\frac{1}{2}} = 2\nu + 1$ where ν is an integer.

Because of the factor $\operatorname{tg} \frac{1}{2}\pi x^{\frac{1}{2}}$ in the expression for the impedance, the impedance becomes infinite when

$$x^{\frac{1}{2}} = 2\nu + 1, \tag{29}$$

where ν is an integer. To obtain the resonance frequencies for a given plasma density, we have to solve equation (26) for s and then introduce equation (29). We find the resonance condition

$$\left.\begin{aligned} s &= x + r \mp (r^2 + 4xr)^{\frac{1}{2}}, \\ x &= (2\nu + 1)^2. \end{aligned}\right\} \tag{30}$$

Equation (30) gives the same solutions as equation (19) when n is odd. The values given by equation (19) when n is even, are not represented in the present case. Consequently, we can use Fig. 10 in the present case if we exchange n^2 for x. The lower branch (Fig. 11), reckoned from the vertical tangent of the curve, then represents the solution obtained using the minus sign in equation (30). The rest of the curve represents the solution of equation (30) with the plus sign. We can conclude that the frequencies of the odd natural modes appear as resonances in the impedance of the system.

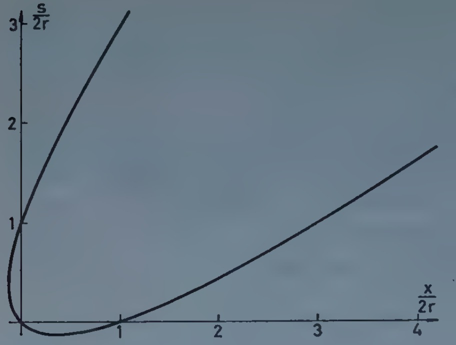

Fig. 11. Graphical representation of equation (30).

As an illustration we give, in Fig. 12, the resonances for the special case when $(2r)^{\frac{1}{2}} = 14$. On the central scale we give the angular frequency divided by Ω and on the lower scale the angular frequency divided by $\pi v/d$. With these scales as reference, the upper line shows the positions of the resonance frequencies. We have plotted one series of resonances above the line and one below. The first series is restricted to the range $\omega > \Omega$. The other begins with seven resonances for imaginary values of ω. Then come five resonances for $0 < \omega < \Omega$. Excluding the first thirteen resonances, all the rest have $\omega > \Omega$.

In any experiment there are losses. This means that the resonances are damped and the damping is more severe the higher the order of the resonance. In the present example it should be easiest to find the first seven resonances above the frequency Ω, shown above the line in Fig. 12. The resonances shown below the line should be as strongly damped as the following resonances above the line—at least. The lower measuring frequency would surely make the damping even more severe.

In an experiment it is usually more convenient to keep the measuring frequency constant and vary the charge density. The asymptotes of Z still correspond to $x_{1,2} = (2\nu + 1)^2$.

$$2r = \frac{[s - (2\nu + 1)^2]^2}{s + (2\nu + 1)^2}. \tag{31}$$

Again we see the similarity with equation (19). The resonances coincide with the natural modes of odd order.

Equation (31) shows that the plasma density for resonance is a single-valued function of the parameter ν.

If we order the resonances according to increasing values of ν, Fig. 13 shows that, to begin with, r decreases with ν, i.e. the plasma density decreases for increasing ν.

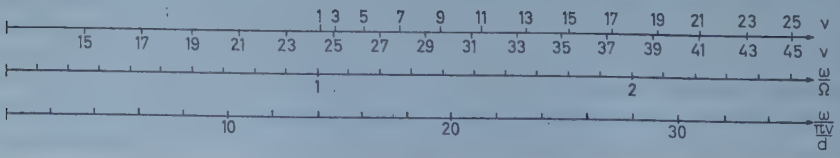

Fig. 12. The resonance frequencies when the resonance conditions are given.

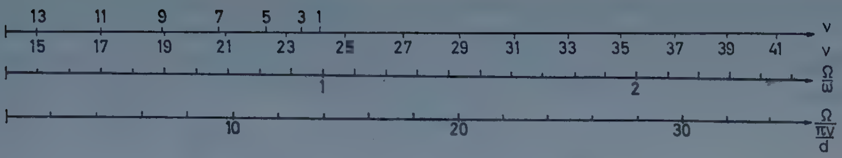

Fig. 13. The resonance values of the plasma density when the frequency of perturbation is given.

The number of resonances with this property equals the smallest integer larger than $\frac{1}{2}(s^{\frac{1}{2}}-1)$. Consequently, no resonances of this character exist if $s<1$. Next, the resonances fall on the part of the curve in figure 9 with negative slope. Now the resonance plasma density increases with the order of the resonance.

The total number of resonances for which $\Omega<\omega$ equals the smallest integer larger than $\frac{1}{2}\,[(3s)^{\frac{1}{2}}-1]$. No resonance with this property exists if $3s<1$.

As an example we choose

$$s^{\frac{1}{2}}=\omega\Big/\frac{\pi v}{d}=14,$$

$$\{\tfrac{1}{2}(s^{\frac{1}{2}}-1)=6\tfrac{1}{2};\quad \tfrac{1}{2}((3\,s)^{\frac{1}{2}}-1)\approx 11.614\}.$$

The resonance of lowest order appears for a value of Ω slightly smaller than ω. Seven resonances of consecutive orders appear for decreasing values of Ω. The next five appear at increasing values of Ω but still have $\Omega<\omega$. Thus, altogether twelve resonances have the property $\Omega<\omega$.

3. Cylindrical geometry

3.1 *Introduction*

It is well known that a homogeneous plasma cylinder of circular cross section has a resonance that can be excited by an oscillating electric field perpendicular to its axis. At resonance the reflected wave is identical with that reflected by a metal cylinder with a diameter of roughly 2/3 of the wavelength. This is true whatever the diameter of the plasma cylinder. Since the value of $r(\partial B/\partial r)\,B^{-1}$ just inside the boundary determines the reflection properties, our statement is true also for an inhomogeneous column provided the density is a function of the radius vector only. We then take for granted that the treatment of the singularity is such that we find no dissipation in the plasma region when the collision frequency is zero.

For a column diameter much smaller than the wavelength it is possible to approximate the solution outside the cylinder by the static field. We then obtain infinite amplitude at resonance. Despite this defect our solution will not differ much from a more stringent solution. In the following we shall use this simplification and turn our attention first to the cylinder with elliptic cross section and then to the problem of an inhomogeneous density distribution. Because the numerical calculations in the latter case are very time-consuming, we have only brought those calculations, which are presented in § 3.3, so far that we see that there is only one resonance.

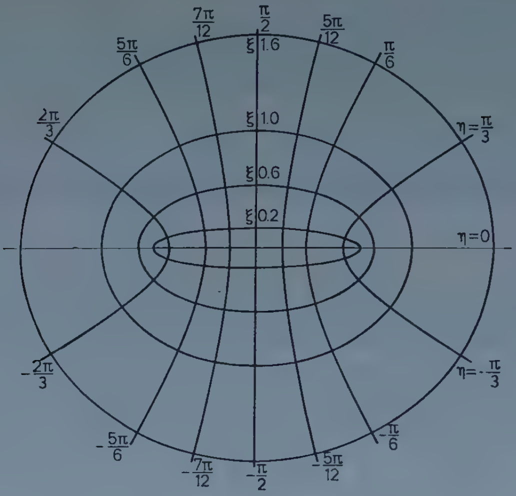

Fig. 14. Elliptical cylinder-coordinates.

3.2. *Scattering by a plasma cylinder of constant density and elliptic cross section*

We shall assume that we have a plasma cylinder of elliptic cross section with constant density inside and vacuum outside. This cylinder is placed in a plane electromagnetic wave propagated in a direction perpendicular to its axis and with the electric vector perpendicular to the axis. If the cross section were circular we should expand the plane wave field outside the boundary in multipole fields and then treat the scattering of each component separately. We proceed similarly in the present case.

The complete solution of the problem requires much tedious work. In order to survey the problem we shall assume that the major axis of the cross section is much smaller than the vacuum wavelength of the exciting field. Thus we change the radiation problem into an electrostatic one, which means that we have to solve the Laplace equation instead of the wave equation. Our problem can be stated in the following way. We have a homogeneous dielectric rod in the form of a straight cylinder with elliptic cross section surrounded by an isotropic medium. At the boundary we prescribe an exciting field of multipole character and wish to find a field that is continuous everywhere inside the cylinder and falls off outside at least as r^{-1}, where r is the distance to the axis. At the boundary the potential and the normal component of the electric displacement must be continuous.

The mathematical treatment will be simplest if we use elliptic coordinates, ξ and η, which are related to the ordinary cartesian coordinates x and y by the defining equations

$$\left.\begin{aligned} x &= c \cosh \xi \cos \eta \qquad (\xi > 0), \\ y &= c \sinh \xi \sin \eta \qquad (0 < \eta < 2\pi). \end{aligned}\right\} \tag{1}$$

As seen in Fig. 14, ξ corresponds to the radius vector in ordinary cylinder coordinates and η corresponds to the angular coordinate. We also see that the curves, $\xi = \text{const.}$, correspond to confocal ellipses, tending to a line of finite and non-zero length when

ξ tends to zero. By a suitable choice of c in equations (1) the boundary of the dielectric cylinder is made to coincide with a curve $\xi = \text{const.}$

Let us denote the scalar potential by Φ. The electric field is then $-\operatorname{grad} \Phi$. We shall assume that the dielectric constant is a tensor (ε) so that the electric displacement is $(\varepsilon)\mathbf{E} = -(\varepsilon) \operatorname{grad} \Phi$.

Since $\operatorname{div} \boldsymbol{D} = 0$, we find

$$\operatorname{div} \{ (\varepsilon) \operatorname{grad} \Phi \} = 0. \tag{2}$$

We easily find that this can be written

$$\frac{\partial}{\partial \xi} \left\{ (\varepsilon) \left(\frac{\partial \Phi}{\partial \xi}, \frac{\partial \Phi}{\partial \eta}, 0 \right) \right\}_{\xi} + \frac{\partial}{\partial \eta} \left\{ (\varepsilon) \left(\frac{\partial \Phi}{\partial \xi}, \frac{\partial \Phi}{\partial \eta}, 0 \right) \right\}_{\eta} = 0, \tag{3}$$

where subscripts ξ and η indicate the ξ and η components of the vector respectively. If there is a magnetic field in the ζ direction but no losses we can write [22]

$$(\varepsilon) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{4}$$

Then equation (3) can be written

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = 0. \tag{5}$$

Attempting a separation of the variables we put

$$\Phi = \mathrm{u}(\xi)\, v(\eta). \tag{6}$$

Since u is a function of ξ alone and v is a function of η alone equation (5) is split into

$$\left. \begin{aligned} u'' - n^2 u &= 0, \\ v'' + n^2 v &= 0. \end{aligned} \right\} \tag{7}$$

Here n^2 is a separation constant.

Since our problem is a physical one we must require that the solution is single valued. The defining equations (1) show that when η has increased by 2π we arrive at the point where we started. We thus require that the solution $v(\eta)$ shall retain its value when η increases by 2π, i.e. n must be an integer, $n = 0, 1, 2, 3 \ldots.$

We require a solution which is continuous and has a continuous derivative across the line $\xi = 0$. This condition is fulfilled if we require the functions u and v be both even or both odd. Hence we get two pairs of solutions

$$\Phi_n = \cosh n\xi \cos n\eta \quad \text{and} \quad \Phi_n = \sinh n\xi \sin n\eta. \tag{8}$$

Outside the cylinder we require a scattered field that decreases with distance. Such fields are

$$\Phi_n = e^{-n\xi} \cos n\eta \quad \text{and} \quad \Phi_n = e^{-n\xi} \sin n\eta. \tag{9}$$

The potential of the exciting fields can be expanded in terms of

$$r^n \cos n(\varphi - \alpha), \tag{10}$$

where r and φ are ordinary cylinder coordinates and α is an arbitrary angle corresponding to a rotation of the field pattern. The expression (10) can be rewritten as

$$r^n \cos n(\varphi-\alpha) = r^n \cos n\varphi \cos n\alpha + r^n \sin n\varphi \sin n\alpha. \tag{11}$$

For a homogeneous field, equations (1) and (11) give

$$r \cos(\varphi-\alpha) = c \cos\alpha \cosh\xi \cos\eta + c \sin\alpha \sinh\xi \sin\eta. \tag{12}$$

This equation shows that the two elementary waves in elliptic coordinates vary sinusoidally with the angle α. Since ξ is constant on the boundary the maximum amplitudes of the $\cos\eta$ and $\sin\eta$ terms are unequal.

The change to elliptic coordinates for higher modes may be made in the following way

$$r^n \cos n\varphi + i r^n \sin n\varphi = (r e^{i\varphi})^n = [c \cosh(\xi + i\eta)]^n = (c/2)^n (e^{\xi+i\eta} + e^{-\xi-i\eta})^n \tag{13}$$

in particular, for $n=2$ and $n=3$.

$$\begin{aligned}
r^2 \cos 2\varphi &= \tfrac{1}{2} c^2 (1 + \cosh 2\xi \cos 2\eta),\\
r^2 \sin 2\varphi &= \tfrac{1}{2} c^2 \sinh 2\xi \sin 2\eta,\\
r^3 \cos 3\varphi &= \tfrac{1}{4} c^3 \cosh 3\xi \cos 3\eta + \tfrac{3}{4} c^3 \cosh\xi \cos\eta,\\
r^3 \sin 3\varphi &= \tfrac{1}{4} c^3 \sinh 3\xi \sin 3\eta + \tfrac{3}{4} c^3 \sinh\xi \sin\eta.
\end{aligned}$$

Thus, each simple mode in cylinder coordinates corresponds to a finite number of simple modes in elliptic coordinates.

Let us assume that the half axes of the cross section of the dielectric rod are a and b $(a > b)$. If $\xi = \xi_0$ is the equation for the boundary we get the defining equation

$$\frac{b}{a} = \operatorname{tangh} \xi_0. \tag{14}$$

The plasma cylinder is situated in a field of the type

$$\Phi = \gamma \cosh n\xi \cos n\eta + \delta \sinh n\xi \sin n\eta. \tag{15}$$

In the special case $n = 1$, the first term corresponds to an electric field with the field vector parallel to the major axis of the cross section of the plasma cylinder and the second term corresponds to a homogeneous electric field with the field vector parallel to the minor axis. Outside the plasma cylinder we get a scattered field of the character

$$\Phi = c_n e^{-n\xi} \cos n\eta + s_n e^{-n\xi} \sin n\eta. \tag{16}$$

Inside the plasma cylinder the field is

$$\Phi = a_n \cosh n\xi \cos n\eta + b_n \sinh n\xi \sin n\eta. \tag{17}$$

The potential Φ and the normal component of the electric displacement are continuous at the boundary of the plasma cylinder. If the (relative) dielectric constant of the medium outside the plasma is ε these two conditions give

$$\left.\begin{aligned} c_n &= [-\{(a-\varepsilon)\,v+b^2\cosh n\xi_0\}\,\gamma+\{-v+(a-\varepsilon)\cosh n\xi_0\}\,b\,\delta]\,S\,e^{n\xi_0}, \\ s_n &= [\{u-(a-\varepsilon)\sinh n\xi_0\}\,b\,\gamma-\{(a-\varepsilon)\,u+b^2\sinh n\xi_0\}\,\delta)\,S\,e^{n\xi_0} \end{aligned}\right\} \quad (18)$$

with

$$S^{-1}=(a\,\mathrm{tgh}\,n\xi_0+\varepsilon)\,(a\coth n\xi_0+\varepsilon)+b^2,$$

$$u=(a\,\mathrm{tgh}\,n\xi_0+\varepsilon)\cosh n\xi_0,$$

$$v=(a\coth n\xi_0+\varepsilon)\sinh n\xi_0.$$

In our paper [22] we computed the elements of the tensor (ε) obtaining

$$\left.\begin{aligned} a &= 1+\sum\frac{n\,e^2}{m\,\varepsilon_0}\left\{\left(\frac{e\,B}{m}\right)^2-\omega^2\right\}^{-1}, \\ b &= \sum\left(\frac{e\,B}{m}\Big/ i\,\omega\right)\frac{n\,e^2}{m\,\varepsilon_0}\left\{\left(\frac{e\,B}{m}\right)^2-\omega^2\right\}^{-1}, \end{aligned}\right\} \quad (19)$$

where we have to sum over all species. In deriving these equations we assumed that the charged particles were free and moved under the influence of the electric field and a static magnetic field, neglecting the influence of perturbations in the magnetic field.

(a) *No magnetic field*

In the case of vanishing magnetic field the solution is simplified considerably. If the exciting electric field is an even function of the coordinates ξ and η, the excited fields are also even functions and similarly for odd functions. In this case the dielectric constant of the plasma cylinder is a scalar constant, ε_i, and the solution is

$$\left.\begin{aligned} a_n &= \frac{\varepsilon\,e^{n\xi_0}}{\varepsilon\cosh n\xi_0+\varepsilon_i\sinh n\xi_0} & b_n &= \frac{\varepsilon\,e^{n\xi_0}}{\varepsilon\sinh n\xi_0+\varepsilon_i\cosh n\xi_0} \\ c_n &= \frac{\tfrac{1}{2}(\varepsilon-\varepsilon_i)\sinh 2n\xi_0\,e^{n\xi_0}}{\varepsilon\cosh n\xi_0+\varepsilon_i\sinh n\xi_0} & s_n &= \frac{\tfrac{1}{2}(\varepsilon-\varepsilon_i)\sinh 2n\xi_0\,e^{n\xi_0}}{\varepsilon\sinh n\xi_0+\varepsilon_i\cosh n\xi_0} \\ &\text{If we assume } \delta=0 \text{ and } \gamma=1. & &\text{If we assume } \gamma=0 \text{ and } \delta=1. \end{aligned}\right\} \quad (20)$$

In an experiment, the field outside the cylinder is most conveniently measured in ordinary cylinder coordinates. To compare our result with experiments we need to convert our solutions to these coordinates.

Generally we have, if we introduce $(x+iy)/c=u$, in order to simplify the notation

$$e^{-\xi-i\eta}=e^{-\operatorname{arcosh} u}=u-(u^2-1)^{\frac{1}{2}}. \quad (21)$$

If we then expand the root in powers of u^{-1} we find

$$e^{-\xi-i\eta}=\tfrac{1}{2}u^{-1}+\tfrac{1}{8}u^{-3}+\tfrac{1}{16}u^{-5}+\tfrac{5}{128}u^{-7}+\tfrac{7}{256}u^{-9}+\dots \quad (22)$$

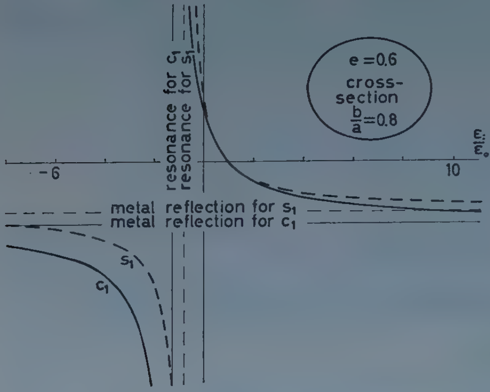

Fig. 15. Reflection properties when the ratio of minor to major axis of the tube cross section is 0.8. Linear scales.

Taking the nth power of this expression we find for $n=2$, 3 and 4

$$\left.\begin{aligned} e^{-2(\xi+i\eta)} &= \tfrac{1}{4}u^{-2}+\tfrac{1}{8}u^{-4}+\tfrac{5}{64}u^{-6}+\tfrac{7}{128}u^{-8}+\ldots, \\ e^{-3(\xi+i\eta)} &= \tfrac{1}{8}u^{-3}+\tfrac{3}{32}u^{-5}+\tfrac{9}{128}u^{-7}+\tfrac{7}{128}u^{-9}+\ldots, \\ e^{-4(\xi+i\eta)} &= \tfrac{1}{16}u^{-4}+\tfrac{1}{16}u^{-6}+\tfrac{7}{256}u^{-8}+\ldots \end{aligned}\right\} \tag{23}$$

We thus find that the scattered field contains an infinite number of elementary solutions in ordinary cylinder coordinates.

For the exciting field

$$\frac{r}{a}\cos(\varphi-\alpha),$$

where we have introduced the major axis of the elliptical boundary in order to make the expression non-dimensional, the scattered field is

$$\frac{\varepsilon-\varepsilon_i}{2}\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}\left[\frac{b\cos\alpha}{\varepsilon_i b+\varepsilon a}e^{-\xi}\cos\eta+\frac{b\sin\alpha}{\varepsilon_i a+\varepsilon b}e^{-\xi}\sin\eta\right], \tag{24}$$

where $e^{-\xi\pm i\eta}$ is obtained from (22) and the conjugate complex equation.

In an experiment we generally keep the frequency of the exciting field constant and vary the density of the plasma. This means that ε_i is our variable and can assume all values from 1 to $-\infty$.

In Fig. 15 c_n and s_n are plotted as functions of $\varepsilon_i/\varepsilon_0$. We find that c_n and s_n vanish when the dielectric constant has the same value inside and outside the boundary. This is self-evident as the dielectric constant then has the same value in all space. When ε_i decreases c_n increases and, for a certain negative value of the ratio $\varepsilon_i/\varepsilon_0$, c_n becomes infinite. This indicates a resonance effect and there is only one resonance for each value of n. The other curve in Fig. 16 shows that the sin $n\eta$ mode is analogous in this respect. The resonances correspond to

$$\frac{\varepsilon_i}{\varepsilon_0}=-\operatorname{ctgh} n\xi \quad\text{and}\quad \frac{\varepsilon_i}{\varepsilon_0}=-\operatorname{tgh} n\xi_0. \tag{25}$$

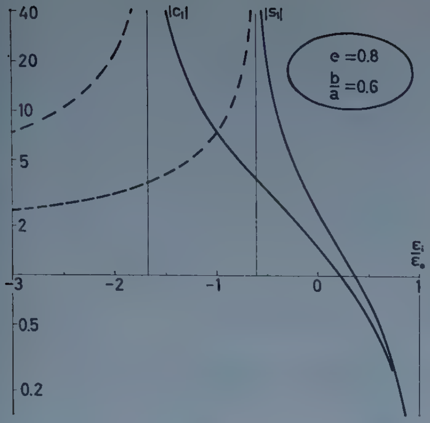

Fig. 16

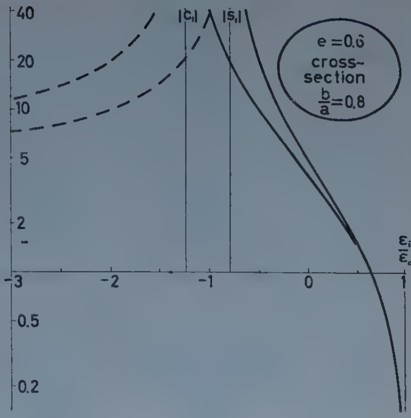

Fig. 17

Fig. 16. Reflection properties when the ratio of minor to major axis of the tube cross section is 0.6.

Fig. 17. Reflection properties when the ratio of minor to major axis of the tube cross section is 0.8. Logarithmic ordinate and linear abscissa.

When ε_i becomes infinite, the plasma cylinder reflects like a metal cylinder of infinite conductivity. The corresponding values of c_n and s_n are

$$(c_n)_{\text{metal}} = -\frac{e^{2n\xi_0}+1}{2}, \qquad (s_n)_{\text{metal}} = -\frac{e^{2n\xi_0}-1}{2}. \tag{26}$$

Introducing these in equation (20) we obtain

$$c_n = (c_n)_{\text{metal}} \frac{-1+\varepsilon_i/\varepsilon_0}{\coth n\,\xi_0 + \varepsilon_i/\varepsilon_0}, \qquad s_n = (s_n)_{\text{metal}} \frac{-1+\varepsilon_i/\varepsilon_0}{\operatorname{tgh} n\,\xi_0 + \varepsilon_i/\varepsilon_0}. \tag{27}$$

In Fig. 16 we notice that the cos $n\eta$ mode is reflected more strongly than the sin $n\eta$ mode. From equation (14) we easily find

$$\operatorname{tgh} n\,\xi_0 = \frac{(a+b)^n-(a-b)^n}{(a+b)^n+(a-b)^n}. \tag{28}$$

When the ellipse does not differ much from a circle, $(a-b)/(a+b)$ is much less than one and we find, approximately

$$\left.\begin{aligned} \operatorname{tgh} n\,\xi_0 &\approx 1-2\left(\frac{a-b}{a+b}\right)^n, \\ \operatorname{ctgh} n\,\xi_0 &\approx 1+2\left(\frac{a-b}{a+b}\right)^n. \end{aligned}\right\} \tag{29}$$

When b/a is much smaller than one, the resonance conditions are simply

$$\left.\begin{aligned} &\varepsilon_i/\varepsilon_0 \approx -n\,a/b \quad \text{for the cos } n\,\eta \quad \text{mode and } n^{\text{th}} \text{ order resonance} \\ &\varepsilon_i/\varepsilon_0 \approx -n\,b/a \quad \text{for the sin } n\,\eta \quad \text{mode and } n^{\text{th}} \text{ order resonance} \end{aligned}\right\} \tag{30}$$

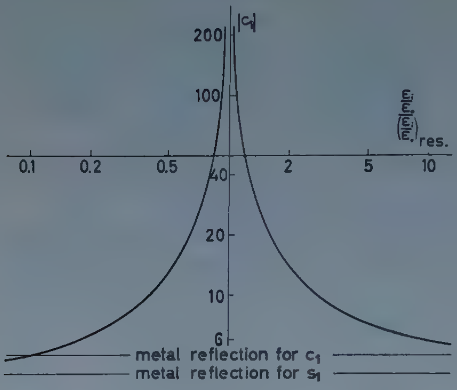

Fig. 18. Normalized resonance curve.

For a circular cylinder there is only one resonance frequency for each value of n. We have found that when the cross section is elliptic each of these resonances is split into two. In Fig. 17 we see that, even for a relatively small deviation from circular cross section, the distance between the two resonances is quite large.

From equation (20) it is evident that

$$c_n(\varepsilon_i, \varepsilon_0) = -s_n(\varepsilon_0, \varepsilon_i), \qquad s_n(\varepsilon_i, \varepsilon_0) = -c_n(\varepsilon_0, \varepsilon_i). \tag{31}$$

Thus the geometric means of the resonance values of $\varepsilon_i/\varepsilon_0$ for the cos $n\eta$ mode and the sin $n\eta$ mode is equal to minus one, i.e. equal to the value for a circular cross section.

$$(\varepsilon_i/\varepsilon_0)_{\cos n\eta} \cdot (\varepsilon_i/\varepsilon_0)_{\sin n\eta} = (-1)^2. \tag{32}$$

Fig. 18 illustrates the same conditions as Fig. 16 but for negative values of $\varepsilon_i/\varepsilon_0$ and with logarithmic axes. The abscissa has been normalized so that the resonance occurs at minus one. The abscissa minus infinity corresponds to metal reflexion. To the right of the resonance in Fig. 18 the curve approaches the resonance level of the sin $n\eta$ mode. It is reached at the plasma frequency. The sign is, however, opposite.

The corresponding curve for the sin $n\eta$ mode is a mirror image of this curve relative to the ordinate axis, normalized to the sin $n\eta$ resonance. Metal reflexion corresponds

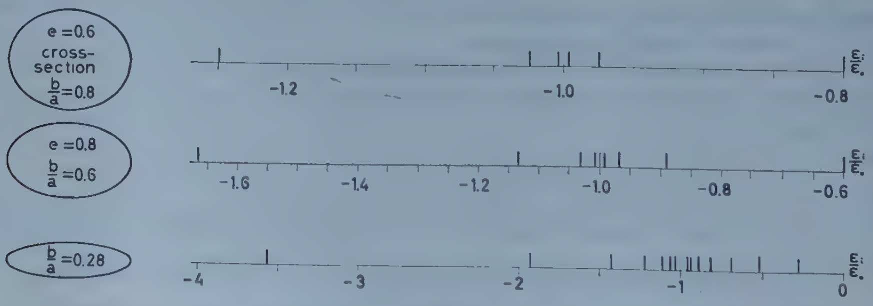

Fig. 19. Distribution of resonances for three values of the eccentricity of the plasma cross section.

Table 1.

n	b/a ...	0.8	0.6	0.28
	(*a*) Resonance for cos $n\eta$ terms. $\varepsilon_i/\varepsilon_0 = f_c(n, b/a)$.			
1		-1.2500	-1.6667	-3.5714
2		-1.0250	-1.1333	-1.9257
3		-1.0028	-1.0318	-1.4330
4		-1.0000	-1.0078	-1.2225
5			-1.0020	-1.1194
	(*b*) Resonance for the sin $n\eta$ terms. $\varepsilon_i/\varepsilon_0 = f_s(n, b/a)$.			
1		0.8000	0.6000	0.2800
2		0.9756	0.8824	0.5193
3		0.9973	0.9692	0.6978
4		0.9997	0.9922	0.8180
5		0.99997	0.9981	0.8934

to the left infinity, whereas at the right infinity, i.e. at the plasma frequency, the reflexion is like the cos $n\eta$ mode for a cylinder of infinite conductivity, again with opposite sign.

Another consequence of the symmetry of equation (31) is that, the reflected energy is equal for the two modes when $\varepsilon_i/\varepsilon_0 = -1$. This is easily seen in Fig. 17.

In Fig. 16 we can see that the cos η mode reflects more energy than a metal cylinder up to a value of $\varepsilon_i/\varepsilon_0$ slightly below zero. We also see that the sin η mode reflects more energy than a metal cylinder even for a slightly positive value of the same ratio.

For a circular cross section all orders of resonance occur at the same value, $\varepsilon_i/\varepsilon_0 = -1$. For an elliptic cross section each resonance splits into two, one above and one below $\varepsilon_i/\varepsilon_0 = -1$. Further resonances of different order occur, at different values of $\varepsilon_i/\varepsilon_0$. Some examples are given in Fig. 19. Thus for an eccentricity of the ellipse of 0.6, we distinguish the dipole, the quadrupole and octopole lines. The higher modes lie too close together to be shown in the figure. When the eccentricity has increased to 0.8 we can distinguish 8 lines and when it has increased to 0.96 at least 10 lines.

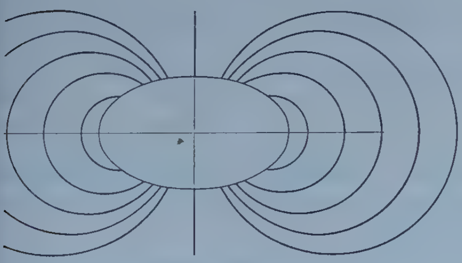

Fig. 20

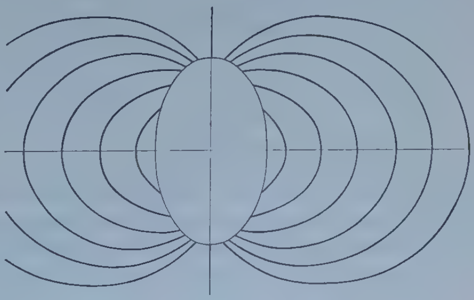

Fig. 21

Fig. 20. Electric field lines of the cos η mode.

Fig. 21. Electric field lines for the sin η mode.

The two dipole modes lie at the extreme left and right and the higher modes appear in turn when we go inwards from these extremes. We also find that the greater the eccentricity the more evenly are the lines distributed. For the highest eccentricity, the dipole resonance occurs nearer to the plasma frequency than to the resonance frequency for circular cross section. Table 1 gives numerical values for the five lowest resonance modes for some values of the eccentricity.

Figs. 20 and 21 illustrate the field pattern for the $\cos\eta$ mode and $\sin\eta$ mode. For the lowest mode the field is homogeneous inside the plasma cylinder. When the eccentricity approaches unity the $\sin\eta$ mode obviously tends towards the plasma resonance of a plane layer. Equation (30) shows that this limit resonance occurs at $\varepsilon_i/\varepsilon_0 = 0$.

(b) *Non-vanishing magnetic field*

We shall assume that the applied frequency is much higher than the gyro frequencies for any ions present, so that the ions may be considered immovable. Introducing $g = eB/m$ and $s = \operatorname{tgh} n\xi_0$ we can rewrite equation (18)

$$\left.\begin{aligned} c_n &= -\left\{\frac{\left(\frac{\Omega^2}{\omega^2}-u_1\right)\left(\frac{\Omega^2}{\omega^2}-u^2\right)}{\left(\frac{\Omega^2}{\omega^2}-v_1\right)\left(\frac{\Omega^2}{\omega^2}-v^2\right)}\gamma + i\,\frac{\varepsilon\,(1+s)\,\frac{\Omega^2}{\omega^2}\,\frac{g}{\omega}}{\left(\frac{\Omega^2}{\omega^2}-v_1\right)\left(\frac{\Omega^2}{\omega^2}-v_2\right)}\delta\right\}\cosh n\xi_0\, e^{n\xi_0}, \\ s_n &= \left\{ i\,\frac{\varepsilon\,(1+s)\,\frac{\Omega^2}{\omega^2}\,\frac{g}{\omega}}{\left(\frac{\Omega^2}{\omega^2}-v_1\right)\left(\frac{\Omega^2}{\omega^2}-v_2\right)}\gamma - s\,\frac{\left(\frac{\Omega^2}{\omega^2}-u_3\right)\left(\frac{\Omega^2}{\omega^2}-u_4\right)}{\left(\frac{\Omega^2}{\omega^2}-v_1\right)\left(\frac{\Omega^2}{\omega^2}-v_2\right)}\delta\right\}\cosh n\xi_0\, e^{n\xi_0}. \end{aligned}\right\} \tag{23}$$

The resonance frequencies are given by

$$v_{1,2} = \frac{1}{2}\left(2 + s\varepsilon + \frac{\varepsilon}{s}\right) \pm \left\{\frac{\varepsilon^2}{4}\left(s - \frac{1}{s}\right)^2 + \frac{g^2}{\omega^2}\,(s+\varepsilon)\left(\frac{1}{s}+\varepsilon\right)\right\}^{\frac{1}{2}} \tag{24}$$

and the zeros by

$$u_{1,2} = \frac{1}{2}(2 - \varepsilon + s\varepsilon) \pm \left\{\frac{\varepsilon^2}{4}(1+s)^2 + \frac{g^2}{\omega^2}(1-\varepsilon)\,(1+s\varepsilon)\right\}^{\frac{1}{2}}, \tag{25}$$

$$u_{3,4} = \frac{1}{2}\left(2 - \varepsilon + \frac{\varepsilon}{s}\right) \pm \left\{\frac{\varepsilon^2}{4}\left(1 + \frac{1}{s}\right)^2 + \frac{g^2}{\omega^2}(1-\varepsilon)\left(1 + \frac{\varepsilon}{s}\right)\right\}^{\frac{1}{2}}. \tag{26}$$

We notice that the scattered wave is elliptically polarized, i.e. there is a non-zero field along both axes of the cross section with a phase difference of $\frac{1}{2}\pi$ radians. The two components generally have different amplitudes.

The resonance conditions are illustrated in Fig. 22. As the surrounding medium is generally vacuum (air), we assume $\varepsilon = 1$. The figure shows that there are two resonances when ω is smaller than the electron gyro frequency, but only when the gyro frequency exceeds ω. In the well-known case of a circular plasma cylinder, the two resonances degenerate into one, which appears when the ordinate is 2. The deriva-

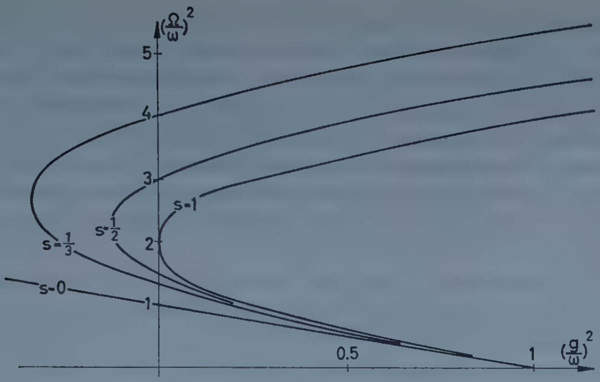

Fig. 22. Resonance conditions for a homogeneous plasma region with elliptical boundary situated in a longitudinal magnetic field. Exciting field parallel to major axis.

tive is infinite at the point when the abscissa is zero. For a tube of greater eccentricity the lower resonance appears at a lower value of the ordinate which must however exceed unity for no magnetic field; in the latter case the curve in the figure is a straight line. The lower resonance is thus limited to the small region between the parabola for a circular cylinder and the straight line for the plane case. The frequency decreases for increasing eccentricity and also for increasing magnetic field.

The higher of the two resonance frequencies is always higher than that corresponding to the parabola of a circular cylinder. In Fig. 23 we see that the ordinate increases for increasing eccentricity and increases for increasing magnetic field. There is no upper limit.

At this point it may be appropriate to stress that a real magnetic field has two effects: one is the one we have explained above but the other one is the change in the distribution of charge. We must therefore be very careful in applying the above calculations to experimental data.

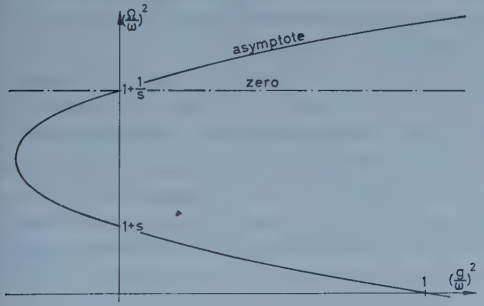

Fig. 23

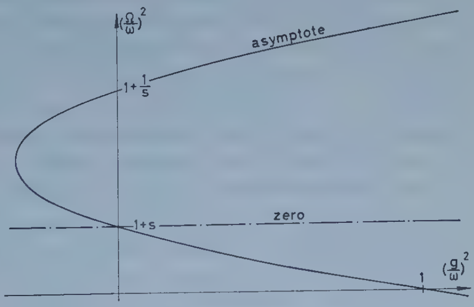

Fig. 24

Fig. 23. The amplitude c_n (along the axis perpendicular to the paper) as a function of Ω^2/ω^2 and g^2/ω^2 for an exciting field parallel to the major axis.

Fig. 24. The amplitude s_n (along the axis perpendicular to the paper) as a function of Ω^2/ω^2 and g^2/ω^2 for an exciting field parallel to the minor axis.

Let us assume a linearly polarized electric field parallel to the minor axis of the cross section of the plasma cylinder, which gives the resonances shown in Fig. 24. The scattered field along the minor axis has a zero for a density that is independent of the strength of the magnetic field. This zero coincides with the highest of the resonance frequencies when the magnetic field vanishes. Consequently there is only one resonance when the magnetic field vanishes but otherwise two resonances with a zero in between. As the component of the scattered field along the major axis has no zero, it has always two resonances. However, its intensity is zero for a vanishing magnetic field.

For an exciting electric field along the major axis there is a zero in the scattered field in the same direction and this zero is independent of the strength of the magnetic field. Its value corresponds to the lower resonance without magnetic field. As before, there is then only one resonance for vanishing magnetic field. For finite value of the magnetic field strength there are two resonances with a zero in between. There is no zero for the field component along the minor axis. However, this component vanishes when the magnetic field strength vanishes.

3.3 *The inhomogeneous plasma cylinder*

In all experiments on the scattering from a plasma cylinder, the plasma has been produced by a gas discharge. It is impossible to obtain a uniform radial distribution of the density in a gas discharge. Therefore, one would like to have a theory for the scattering from an inhomogeneous plasma cylinder. The main difficulty has been and is still the singular surface where the dielectric constant vanishes. To introduce a step function for the density or a collision frequency that is finally put equal to zero only serves to hide the difficulty. The first method leads to one resonance frequency for each step introduced. Actually there should be only one resonance. The second method leads to losses also in the limit when the collision frequency is zero.

In order to resolve the difficulty, we must first recognize that the concept of a dielectric constant is not applicable at and near the singular surface. According to the simple theory, the electric field strength approaches infinity at the singularity. When defining a dielectric constant one assumes that the relative change in the macroscopic fields is small within distances typical for the microscopic structure of the medium. This condition is not fulfilled near the singularity and we cannot use the concept of a dielectric constant in this region. Thus we must ask: How shall we treat the problem at and near the singularity? One possibility is to introduce particles distributed at random in space with a certain mean density and take account of the fields created by each of them. This, however, seems to be too difficult to be feasible.

For a continuous medium we find that $\Delta(\partial E/\partial r)/E$ increases to an infinite value at the singular surface (r is the coordinate along the density gradient and Δ the interelectronic distance). When the change in the field in the distance Δ is comparable to the field itself, i.e. when $\Delta(\partial E/\partial r)/E$ is of the order unity, the idealization of a continuous medium must fail. The region where the concept of a dielectric constant is invalid has a width of roughly 2Δ and thus contains very few particles. We may consider this region to be a vacuum as this cannot introduce any significant error. We can show that the position of the boundary between the plasma and the vacuum region is not critical. Despite the fact that such a treatment is far from strict, it will help us to survey the problem of the inhomogeneous plasma.

We shall idealize the physical problem and state it mathematically in the following way. Let us introduce two surfaces, one on each side of the singular surface, and at a distance Δ_1 from it. As ε is zero on the singular surface the dielectric constant has the same numerical value but opposite sign on these two surfaces. Between the surfaces there is a vacuum and outside the plasma is undisturbed and assumed to be a continuous medium. The potential and the normal component of the electric displacement shall be continuous at the two surfaces. Then we let Δ_1 tend to zero. In the limit we obtain a well-defined mathematical problem.

We shall assume a plasma in the form of a circular cylinder with the density being a function of the radius only. We also assume a plane electromagnetic wave in the surrounding medium, whose direction of propagation and electric vector are both perpendicular to the axis of the cylinder. This field can be expressed as a sum of multipole fields. In this paper we shall investigate the influence on the plasma cylinder of the multipole fields. To simplify the problem we shall assume that the radius of the tube is much smaller than the wavelength of the exciting field. It is then possible to use an electrostatic approximation, thus: We have a dielectric rod of circular cross section and infinite length and with a dielectric constant which is a function only of the distance from the axis. It is placed in a multipole field. In this approximation all fields are assumed to be independent of time. Since Maxwell's equations give

$$\operatorname{curl} \boldsymbol{E} = 0, \tag{1}$$

we can introduce a scalar potential so that

$$\boldsymbol{E} = \operatorname{grad} \Phi. \tag{2}$$

Further $\operatorname{div} \boldsymbol{D} = 0$ and $\boldsymbol{D} = \varepsilon \varepsilon_0 \boldsymbol{E}$. (3)

From equations (2) and (3) we obtain

$$\operatorname{div} (\varepsilon \operatorname{grad} \Phi) = 0. \tag{4}$$

If we introduce cylinder coordinates, equation (4) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\varepsilon\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \vartheta}\left(\frac{\varepsilon}{r}\frac{\partial \Phi}{\partial \vartheta}\right) = 0. \tag{5}$$

To separate the variables, put

$$\Phi = R(r)\,\Theta(\vartheta). \tag{6}$$

Then equation (5) is split into equations (7) and (8), where n^2 is a separation constant

$$r\frac{d}{dr}\left(r\varepsilon\frac{dR}{dr}\right) - n^2 \varepsilon R = 0, \tag{7}$$

$$\frac{d^2}{d\vartheta^2}\Theta + n^2\Theta = 0. \tag{8}$$

In general the plasma is densest on the axis and its density decreases outwards. This means that the dielectric constant increases outwards. Let us study the case where the dielectric constant varies linearly with the distance from the axis.

$$\varepsilon = b\,r - a. \tag{9}$$

If we introduce this expression into equation (7) we find

$$r^2\,(b\,r-a)\,\frac{d^2 R}{d\,r^2} + r\,(2\,b\,r-a)\,\frac{d\,R}{d\,r} - n^2\,(b\,r-a)\,R = 0. \tag{10}$$

This equation has three singularities with the exponents

$$\left.\begin{array}{l} n \quad \text{and} \quad -n \quad \text{at } r=0, \\ 0 \quad \text{and} \quad 0 \quad \text{at } r=a/b, \\ \mu = \dfrac{1}{2} \pm \left(n^2 + \dfrac{1}{4}\right)^{\frac{1}{2}} \quad \text{at } r=\infty. \end{array}\right\} \tag{11}$$

The solution can be expressed using hypergeometric series. At $r=0$ and at $r=a/b$ the difference between the characteristic exponents is integral. Hence only one of the linearly independent solutions can be expressed as a power series. At these two points the other linearly independent solution contains a logarithmic term. At infinity, however, both the linearly independent solutions are expressible as power series in $1/r$.

We want a solution that is finite at $r=0$ and choose the one without the logarithmic term. We then have a solution that is determined for $r<a/b$.

At $r=a/b$ this solution increases logarithmically. For large values of r our solution can be expressed as a sum of two linearly independent hypergeometric series of the variable $1/r$. Also this solution increases logarithmically when we approach $r=a/b$. As explained earlier, we now have to determine the coefficients of the two hypergeometric series in such a way that the two solutions, the one outside and the one inside the singularity, approach the same value when r tends to a/b. The derivatives shall approach the same numerical value but have opposite sign on opposite sides of the singularity. This can be written as follows

$$\left.\begin{array}{l} \lim\limits_{\Delta\to 0}\,(\Phi_{r=a/b+\Delta} - \Phi_{r=a/b-\Delta}) = 0, \\ \lim\limits_{\Delta\to 0}\left(\left(\dfrac{\partial\Phi}{\partial r}\right)_{r=a/b+\Delta} + \left(\dfrac{\partial\Phi}{\partial r}\right)_{r=a/b-\Delta}\right) = 0. \end{array}\right\} \tag{12}$$

The singularity will have its standard position if we introduce the new independent variable

$$\varrho = b\,r/a. \tag{13}$$

In order to get one of the characteristic exponents at the origin equal to zero, we also introduce a new dependent variable

$$\Psi = R\,\varrho^{-n}. \tag{14}$$

Then the differential equation takes the standard form

$$(1-\varrho)\frac{d^2\psi}{d\varrho^2}+\{2n+1-(2n+2)\varrho\}\frac{d\psi}{d\varrho}-n\psi=0. \tag{15}$$

The parameters of this equation are

$$a=2n+\tfrac{1}{2}+\delta,$$

where $$\delta=(n^2+\tfrac{1}{4})^{\frac{1}{2}}-n;\quad a\,b=n,\quad b=\tfrac{1}{2}-\delta,\quad c=a+b=2n+1. \tag{16}$$

The solution of equation (15) that is regular at $\varrho=0$, the hypergeometric series, is

$$F(a,b;\,a+b;\,\varrho), \tag{17}$$

$$\frac{dF}{d\varrho}=\frac{n}{2n+1}F(a+1,\,b+1;\,a+b+1;\,\varrho). \tag{18}$$

The parameters a and b are positive when the dielectric constant is negative on the axis of the tube. In this case ϱ is positive. If the dielectric constant is positive in the whole plasma region, a is negative but b positive, i.e. ϱ is negative. Thus we also need a solution of our equation for $\varrho<0$. It is more suitable to express this solution in the hypergeometric series of the variable $1/(1-\varrho)$. We have

$$F(a,b;\,a+b;\,\varrho)=\frac{B_1}{(1-\varrho)^a}F\left(a,a;\,a-b+1;\,\frac{1}{1-\varrho}\right)+$$
$$+\frac{B_2}{(1-\varrho)^b}F\left(b,b;\,b-a+1;\,\frac{1}{1-\varrho}\right) \tag{19}$$

with the coefficients

$$\left.\begin{aligned}B_1&=\Gamma(b+a)\,\Gamma(b-a)\,\{\Gamma(b)\}^{-2},\\ B_2&=\Gamma(a+b)\,\Gamma(a-b)\,\{\Gamma(a)\}^{-2}.\end{aligned}\right\} \tag{20}$$

In order to get a solution on the other side of the singularity we retain the dominating terms of the solution $R=\varrho^n F(a,b;\,a+b,\,\varrho)$ near the singularity. We then obtain

$$R=\varrho^n\{h_0-\log(1-\varrho)\}/B(a,b)+\ldots\quad\text{with}\quad h_0=2\psi(0)-\psi(a)-\psi(b), \tag{21}$$

$$\frac{dR}{d\varrho}=\frac{1}{B(a,b)}\Bigl(\frac{1}{1-\varrho}+1+(1-\varrho)(\ldots). \tag{22}$$

Near infinity the solution can be expressed in two linearly independent hypergeometric series. This fact can be used to obtain a solution when ϱ exceeds unity. We have

$$R=\frac{B(a,1-b)}{B(a,b)}\,p\varphi=\frac{B(b,1-a)}{B(a,b)}\,q\psi_1, \tag{23}$$

where p and q are arbitrary constants to be determined and φ and ψ are defined as follows

$$\varphi = \varrho^{n-a} F(a,\ 1-b;\ 1+a-b;\ \varrho^{-1}), \tag{24}$$

$$\psi_1 = \varrho^{n-b} F(b,\ 1-a;\ 1+b-a;\ \varrho^{-1}) \tag{25}$$

and their derivatives are

$$\frac{d\varphi}{d\varrho} = \frac{n-a}{\varrho}\,\varphi - \frac{\varrho^{n-a-2}}{2} F(1+a,\ 2-b;\ 2+a-b;\ \varrho^{-1}), \tag{26}$$

$$\frac{d\psi_1}{d\varrho} = \frac{n-b}{\varrho}\,\psi_1 - \frac{\varrho^{n-b-2}}{2} F(1+b,\ 2-a;\ 2+b-a;\ \varrho^{-1}). \tag{27}$$

For determining p and q we use the dominating terms of R and $dR/d\varrho$ when ϱ is near unity. We find

$$\varphi \approx \frac{h_1 - \log(\varrho-1) + (\varrho-1)(\ldots)}{B(a, 1-b)} \quad \text{with} \quad h_1 = 2\psi(1) - \psi(a) - \psi(1-b), \tag{28}$$

$$\frac{d\varphi}{d\varrho} \approx \frac{1}{B(a, 1-b)} \left\{ -\frac{1}{\varrho-1} + 1 + (\varrho-1)(\ldots) \right\}, \tag{29}$$

$$\psi_1 = \frac{h_2 - \log(\varrho-1) + (\varrho-1)(\ldots)}{B(b, 1-a)} \quad \text{with} \quad h_2 = 2\psi(1) - \psi(b) - \psi(1-a), \tag{30}$$

$$\frac{d\psi_1}{d\varrho} = \frac{1}{B(b, 1-a)} \left\{ -\frac{1}{\varrho-1} + 1 + (\varrho-1)(\ldots) \right\}. \tag{31}$$

As mentioned before, p and q are to be determined from the conditions that R shall be continuous at $\varrho = 1$ and that the derivative of R shall approach the same absolute value but with opposite sign when we approach $\varrho = 1$ from the two sides. Thus

$$\left.\begin{aligned} &\lim_{\Delta\to 0} \{R(1-\Delta) - R(1+\Delta)\} = 0, \\ &\lim_{\Delta\to 0} \left\{ \left(\frac{dR}{d\varrho}\right)_{\varrho=1-\Delta} + \left(\frac{dR}{d\varrho}\right)_{\varrho=1+\Delta} \right\} = 0. \end{aligned}\right\} \tag{32}$$

If we introduce the dominating terms of R and $dR/d\varrho$ we find

$$\left.\begin{aligned} p &= \frac{h_0 - h_2}{h_1 - h_2} = \tfrac{1}{2}, \\ q &= \frac{h_1 - h_0}{h_1 - h_2} = \tfrac{1}{2}. \end{aligned}\right\} \tag{33}$$

The case $a=0$

In this case our equation is reduced to

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n^2R=0. \tag{34}$$

The solution that is finite at origin is

$$R=A\,r^{-\frac{1}{2}+n+\delta_n}, \tag{35}$$

$$\frac{dR}{dr}=\left(-\tfrac{1}{2}+n+\delta_n\right)A\,r^{-\frac{3}{2}+n+\delta_n}, \tag{36}$$

where A is an arbitrary constant.

A first order equation of the problem

At the boundary between the plasma cylinder and the vacuum region, the tangential component of the electric field strength must be continuous, which means that R must be continuous. The normal component of the electric displacement, i.e. $\varepsilon\,dR/dr$, is also continuous. If we then introduce the new variable

$$\eta=\frac{\varrho\varepsilon}{aR}\frac{dR}{d\varrho}, \tag{37}$$

we know that it must be continuous at the boundary of the plasma cylinder. Using (9), (13) and (37), equation (10) can be transformed into a Riccati equation

$$\frac{d\eta}{d\varrho}=n^2\frac{\varrho-1}{\varrho}+\frac{\eta^2}{\varrho(1-\varrho)}. \tag{38}$$

We want a solution of (10) that is finite near the origin. If we write this solution as a power series, retaining only the first two terms, we obtain

$$R\approx\varrho^n+\frac{n}{2n+1}\varrho^{n+1}, \tag{39}$$

$$\frac{dR}{d\varrho}\approx n\varrho^{n+1}+\frac{n(n+1)}{2n+1}\varrho^n. \tag{40}$$

Introducing this in the definition of η, we find

$$\eta\approx-n\frac{1-\{n/(2n+1)\}\varrho+\dots}{1+\{n/(2n+1)\}\varrho+\dots}, \tag{41}$$

$$\frac{d\eta}{d\varrho}\approx\frac{2n^2}{2n+1}\,\frac{1+\dots}{1+\dots}. \tag{42}$$

In this way we find the requirements at the origin for the solution of the Riccati equation

$$\left.\begin{aligned} \eta &= -n \\ \frac{d\eta}{d\varrho} &= \frac{2n^2}{2n+1} \qquad\qquad (\text{at } \varrho = 0) \\ \frac{d^2\eta}{d\varrho^2} &= \frac{n^2}{(n+1)(2n+1)^2} \end{aligned}\right\} \tag{43}$$

For large values of $|\varrho|$ it may be more suitable to use its inverse value $\vartheta = \varrho^{-1}$. Then equation (38) becomes

$$\frac{d\eta}{d\vartheta} = n^2 \frac{\vartheta - 1}{\vartheta^2} - \frac{\eta^2}{\vartheta - 1}. \tag{44}$$

The boundary value problem

Inside a circle of radius r_0 the medium is described by equation (9) and outside r_0 there is vacuum. In the latter region we assume a source with $R = (r/r_0)^n$ and a field of the character $R = \alpha (r_0/r)^n$, where α is an arbitrary constant. Hence the field in the vacuum region is

$$R = \left(\frac{r}{r_0}\right)^n + \alpha \left(\frac{r_0}{r}\right)^n; \quad r > r_0. \tag{45}$$

At the boundary the normal component of the electric displacement, i.e. $\varepsilon (dR/dr)$, or equivalently η, must be continuous. The tangential component of the electric field must also be continuous, i.e. R must be continuous.

$$\eta = \frac{n\,\varepsilon_{r_0+0}}{-\varepsilon_{r=0}} \frac{1 - \alpha\,(r_0/r)^{2n}}{1 + \alpha\,(r_0/r)^{2n}} \quad \text{when} \quad r > r_0. \tag{46}$$

At the boundary we find

$$\eta_{r_0-0} = \frac{n\,\varepsilon_{r_0+0}}{-\varepsilon_{r=0}} \frac{1-\alpha}{1+\alpha}, \tag{47}$$

where η_{r_0-0} is a solution of equation (38). Hence

$$\alpha = \left(1 + \frac{\varepsilon_{r=0}}{\varepsilon_{r_0+0}} \frac{\eta_{r_0-0}}{n}\right) \left(1 - \frac{\varepsilon_{r=0}}{\varepsilon_{r_0+0}} \frac{\eta_{r_0-0}}{n}\right)^{-1}. \tag{48}$$

In the special case where the absolute variation of the dielectric constant from axis to boundary is much smaller than unity we can use an approximate value of η at the boundary. Using the approximation (43) and remembering that $\varepsilon = a(\varrho - 1)$, we find

$$\eta_{r_0-0} = -n + \frac{2n^2}{2n+1}\left(1 - \frac{\varepsilon_{r_0-0}}{\varepsilon_{r=0}}\right) + \ldots = -\frac{n}{2n+1} - \frac{2n^2}{2n+1}\frac{\varepsilon_{r_0-0}}{\varepsilon_{r=0}} + \ldots \tag{49}$$

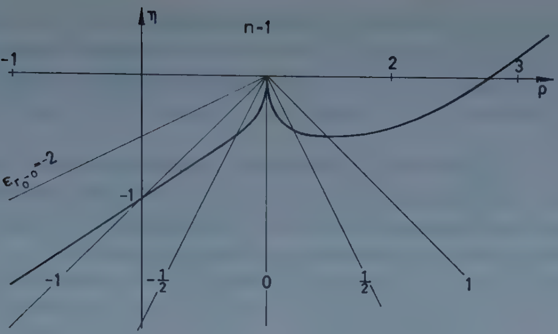

Fig. 25. The dipole resonance condition for different values of the dielectric constant just inside the boundary.

and consequently
$$\alpha \approx \frac{\varepsilon_{r_0+0}-\varepsilon_{r=0}+2n\,(\varepsilon_{r_0+0}-\varepsilon_{r_0-0})}{\varepsilon_{r_0+0}+\varepsilon_{r=0}+2n\,(\varepsilon_{r_0+0}+\varepsilon_{r_0-0})}. \tag{50}$$

If the column is homogeneous, expression (50) assumes the familiar form

$$\alpha=\frac{\varepsilon_{r_0+0}-\varepsilon_{r_0-0}}{\varepsilon_{r_0+0}+\varepsilon_{r_0-0}}. \tag{51}$$

When the dielectric constant is continuous at the boundary, the expression (50) has the form

$$\alpha=\frac{\varepsilon_{r_0+0}-\varepsilon_{r=0}}{(4n+1)\,\varepsilon_{r_0+0}+\varepsilon_{r=0}}. \tag{52}$$

In this case α is still proportional to the difference between the dielectric constant outside the cylinder and its value at the origin. Because of the factor $4n+1$ in the denominator, α is relatively insensitive to $\varepsilon_{r=0}$.

Resonance

Assume a plasma cylinder situated in a medium of dielectric constant ε_{r_0+0}. Equations (9) and (13) give for the value of ϱ at the boundary

$$\varrho=1-\varepsilon_{r_0-0}/\varepsilon_{r=0}. \tag{53}$$

At resonance we further require that α shall be infinite. Then equation (48) gives

$$\eta=n\,\varepsilon_{r_0+0}/\varepsilon_{r=0}. \tag{54}$$

Eliminating $\varepsilon_{r=0}$ between equation (53) and (54) we find

$$\eta=n\,(\varepsilon_{r_0+0}/\varepsilon_{r_0-0})\,(1-\varrho). \tag{55}$$

This is a straight line as seen in Fig. 25. The dielectric constant just inside the boundary is constant for each line but $\varepsilon_{r=0}$ varies along the line. We assume that the dielectric constant of the outer medium, ε_{r_0+0}, is positive. Equation (54) then shows that η is negative when the dielectric constant at the origin is negative.

The straight line on the extreme right corresponds to a plasma density equal to zero at the boundary. For increasing density at the boundary the line becomes steeper and the vertical line corresponds to $\varepsilon = 0$ at the boundary. In the next quadrant ε is negative at the boundary. The cusped curve in Fig. 25 corresponds to the solution of equation (38).

For $\varepsilon_{r=0} > 0$, i.e. $\eta > 0$ there is no intersection between any of these straight lines and the solution of equation (38). Thus, when the dielectric constant is positive at the origin there exists no resonance for a cold plasma with constant density gradiant in radial direction, assuming the density to decrease outwards.

We have assumed vacuum outside the cylinder. For vanishing plasma density at the boundary the resonance then occurs when $\varrho = 1.40$, i.e. when the dielectric constant at origin is -2.5. This intersection occurs at the minimum on the curve of the solution of equation (38). This means that for increasing values of the plasma density at the boundary the crossing occurs at greater values of η and equation (54) shows that $\varepsilon_{r=0}$ must decrease (both η and $\varepsilon_{r=0}$ are negative quantities). In other words, to remain at resonance when we increase the plasma density at the boundary we must also increase the density at the origin. When the dielectric constant has reached the value zero at the boundary the plasma density at the origin must be infinite.

Here we must add that a cylinder with infinite density does not show any resonance. The point corresponding to $\varepsilon = 0$ at the boundary is thus a singular point which we can exclude from further discussion.

When further increasing the density at the boundary, we have to decrease the density at the origin in order to retain resonance and $\varepsilon_{r_0-0} = -1$ corresponds to a homogeneous plasma (the intersection is at $\varrho = 0$). Up to this point the intersections have been in the right half plane in the figure, corresponding to a plasma density increasing inwards. If we increase the density at the boundary further, the intersections fall in the left half plane, corresponding to a plasma that is densest at the boundary. However, the density at the origin must be larger than that corresponding to $\varepsilon_{r=0} = 0$.

Thus, for a given plasma density at the boundary we find one and only one value of the density at the origin that gives resonance.

Let us now assume that the ratio of the densities at the boundary and at the origin is constant. As a measure of the density we use the plasma frequency Ω. If Ω_0 is the value at the origin and Ω_b the value at the boundary we put

$$\frac{\Omega_b}{\Omega_0} = k = \text{positive constant.} \tag{56}$$

Then equations (53) and (55) can be written as

$$\varrho = (1 - k^2)\,\Omega_0^2\,(\Omega_0^2 - \omega^2)^{-1}, \tag{57}$$

$$\eta = n\,\varepsilon_{r_0+0}\,(1 - \varrho\,(1 - k^2)^{-1}). \tag{58}$$

We again find a linear relationship between η and ϱ. All lines pass through the point $\eta = n\,\varepsilon_{r_0+0}$ and $\varrho = 0$. When the outer medium is vacuum, this point corresponds to a plasma with the density zero. The parts above this point correspond to $\varepsilon > 0$ in the plasma. Since these parts do not intersect the curve corresponding to the solution of equation (38), they are not shown in Fig. 26. The points corresponding to $\Omega_0 = \omega$

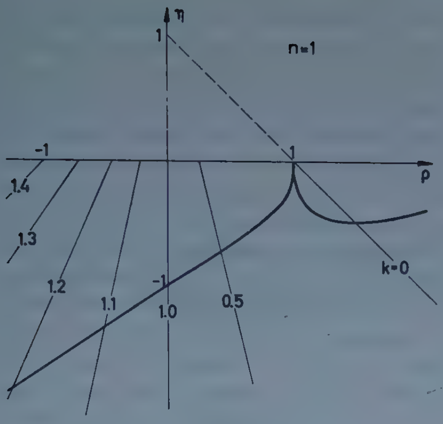

Fig. 26

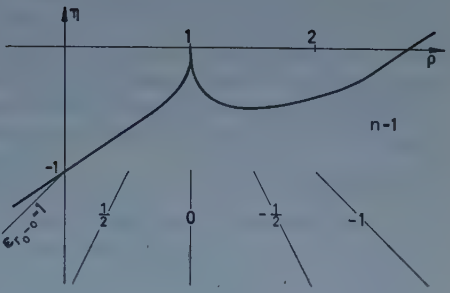

Fig. 27

Fig. 26. The dipole resonance condition for different values of the ratio, k, between the plasma density at the boundary and at the origin.

Fig. 27. The condition for vanishing reflected wave for different values of the dielectric constant just inside the boundary.

lie at $\eta = -\infty$ and for increasing density we move upwards along the lines reaching $\eta = 0$ for infinite density. The line for zero density at the boundary is on the extreme right in Fig. 26 and is identical with the corresponding curve in the previous figure.

When k is positive and smaller than a certain limit (about 0.44 for $n = 1$ and 0.34 for $n = 2$) there are three intersections. This means that if we increase the density while keeping k constant, we find three resonances. If k is above that limit there is only one intersection and hence only one resonance.

The vertical line $k = 1$, corresponds to a homogeneous plasma. The lines to the left correspond to a plasma that is densest at the boundary.

Vanishing "reflected wave"

The constant α vanishes when

$$\eta = -n\,\varepsilon_{r_0+0}/\varepsilon_{r=0}. \tag{59}$$

If we assume $\varepsilon_{r_0+0} > 1$ and $\varepsilon_{r=0} < 1$ the region $-n\,\varepsilon_{r_0+0} < \eta < 0$ does not correspond to any physical situation of the types considered here. Eliminating $\varepsilon_{r=0}$ as before, we find (see Fig. 27)

$$\eta = n\,(\varepsilon_{r_0+0}/\varepsilon_{r_0-0})\,(\varrho - 1) \tag{60}$$

when the density is given on both sides of the boundary. Under the conditions given here no solutions can be found. Under less stringent conditions solutions are possible.

Similarly, we find (see Fig. 28)

$$\eta = n\,\varepsilon_{r_0+0}\,\{\varrho\,(1-k^2)^{-1} - 1\} \tag{61}$$

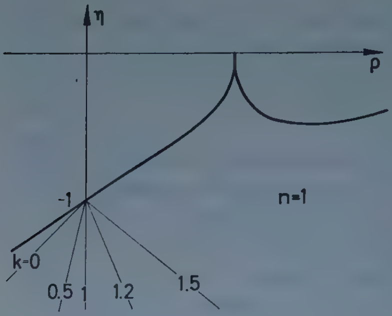

Fig. 28. The condition for vanishing reflected wave for different values of the ratio, k, between the plasma density at the boundary and at the origin.

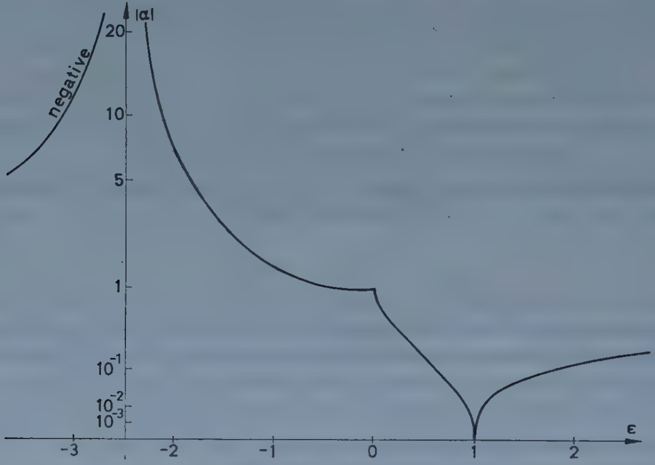

Fig. 29. The "reflection coefficient" for the dipole mode as a function of the dielectric constant at the origin when the plasma density vanishes at the boundary.

when the ratio of the density at the boundary and at the origin is given. There is no value of k such that the reflected wave vanishes. Under less stringent conditions solutions are possible.

If we plot α as a function of $\varepsilon_{r=0}$ when the density at the boundary is zero and there is vacuum outside, the result is that given in Fig. 29. We find $\alpha = 0$ when $\varepsilon_{r=0} = 0$ as this corresponds to vacuum everywhere. Then α increases when ε decreases. (The derivative at $\varepsilon = 1$ is finite; the vertical tangent in the figure is due to the non-linear scale.) At $\varepsilon = 0$ there is a discontinuity in the slope but α remains finite, non-zero and continuous. Resonance occurs for $\varepsilon \approx -2.5$ and when ε decreases further, α decreases and finally approaches unity.

4. Measurements on plasma resonators

L. Tonks [1] seems to have been the first to publish experimental evidence for the existence of several resonances of dipole character. Two peaks are clearly seen in his curves.

Tonks used a gas discharge in mercury vapour in a glass tube of circular cross section. The discharge tube was placed between parallel condenser plates, which were connected to a Lecher line. He measured the reactive and the resistive parts of the impedance due to the presence of the plasma as a function of the plasma density. The measurement was made for different values of the temperature of the condensed mercury (from 13°C to 45°C), for different values of the frequency (from 159 to 366 Mc/s) and for different sizes of the condenser plates between which the discharge tube was placed. Nine measurements are tabulated in the paper. Tonks mentioned that there is also a third resonance in the cases when "the applied field was decidedly non-uniform".

The next paper of interest in this connection was published 20 years later by D. Romell [4]. He used a gas discharge tube of the same radial extension and filled with mercury vapour. The curves refer to a case when the temperature of the condensed mercury is about 25°C. The measurement was made at a frequency of 10^9 c/s. Romell used a radiating antenna at 1.5 m distance from the tube and measured the reflected signal at about the same distance. The published curve shows three resonances.

The third paper of interest was published by A. Dattner [5]. He used a tube having a diameter 1/5 of that used by the other experimenters and at the same time a frequency 5 times that used by Romell. Further, he placed the discharge tube in a rectangular wave guide, operating in the TE_{01} mode, in such a way that the electric field was perpendicular to the axis of the tube. He measured the energy passing the cross section where the discharge tube was placed. At resonance, only part of the energy of the incident wave passed the section containing the discharge tube. In the published curve at least six resonances can be seen. Also in this case the gas was mercury vapour but the temperature of the condensed mercury is not given.

Extensive experimental data have already been published, but more are needed if we want detailed information about the influence of the different parameters and, before a quantitative comparison between theory and experiment is possible, the theory has to be refined.

Our theory shows that the product of the frequency and the distance between the reflecting plates is an important parameter if we assume that the electron temperature is constant. In the experiments performed by Romell and Dattner the mean free path of the electrons, corresponding to the temperature of the condensed mercury is several times the diameter of the tube. This means that we should expect a large number of resonances. In the other measurements the mean free path is only somewhat larger than the diameter of the tube. Therefore, we should expect fewer resonance peaks, which agrees with the measurements.

In Fig. 30 the abscissa is the product of the frequency and the diameter of the discharge tube. The ordinate is the ratio of the density at the second maximum and that at the major maximum. The measured points are grouped along the upper line. Along the lower line we have plotted the ratio of the density at the tertiary maximum and that at the major maximum. The result seems to indicate that the positions of the minor maxima are determined by the value of fd. Despite the fact that the gas density, the linear dimension and the frequency used have been varied roughly a power of ten, the spread of the points in the figure is not very large. It therefore appears that the influence of the mean free path on the positions of the minor maxima is a second-order effect.

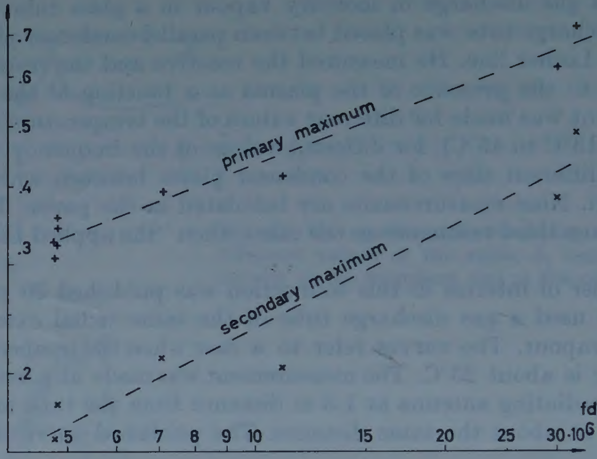

Fig. 30. A collection of experimental data on the relative position of resonances as a function of the quantity $f\,d$.

Qualitatively the result is what we should expect from our theory, but at present we cannot make a quantitative comparison.

Tonks, 1931 [1] published some measurements on a tube of elliptical cross section. His curves show two resonances for the major axis and only one for the minor axis. The experiment thus seems to indicate that when we deform a circular cross section each dipole resonance is split into two.

Dattner has made some measurements on a plasma resonator with elliptical cross section. The tube was placed in a rectangular wave guide in such a way that the electric field was perpendicular to the axis of the tube. The measurements were made at a frequency of 5×10^9 c/s. From the published data we can clearly see that the resonance pattern is different for the case when the major axis of the cross section is along the electric field and when the minor axis of the cross section is along the field. We can easily see that the plasma density necessary for the main resonance is larger in the first of these cases.

The results of some measurements made with Dattner's equipment are shown in Fig. 31. The abscissa is proportional to the electron density in the tube and the ordinate is proportional to the transmitted energy in the wave guide. When the angle between the electric field and the major axis of the cross section is zero or $\frac{1}{2}\pi$ we find the curves that Dattner reported. For intermediate values of the angle we find a weighted mean of these two curves. This may be interpreted thus. There is one resonance pattern corresponding to each axis of the cross section and the excitation of these two patterns is proportional to the component of the electric field along the respective axes.

We have discussed the case of elliptic cross section and homogeneous density with the tube situated in free space. In the measurements the density was not homogeneous and the tube was not situated in free space. Therefore, a quantitative comparison between theory and experiment is not fully appropriate. Theory and experiment agree in that we have one set of resonances for each of the axes of the cross section and in that a larger density is necessary for resonance when the major axis is parallel

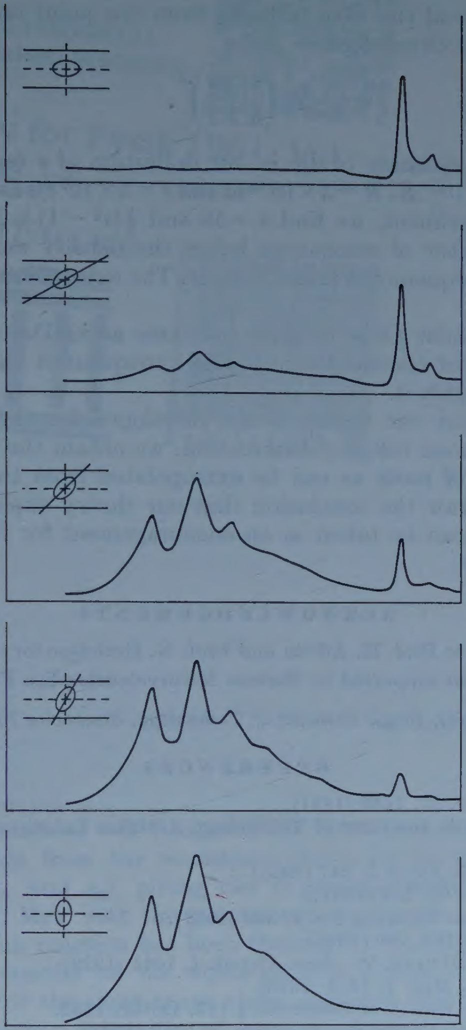

Fig. 31. The measured value of the energy passing the cross section in a wave guide containing a plasma column of elliptical cross section, as a function of the angle between the electric field and the major axis of the plasma cross section.

to the electric field. The major axis gives a more pronounced resonance than the minor axis—as the theory predicts. The order of magnitude is correct.

If we wish to make a rough comparison between theory and experiment we may take the measurements made by Dattner and those made by Romell on tubes with circular cross section. We may call the radius R. The mean distance between two points on the circumference is $4R/\pi$. For a Maxwellian electron-velocity distribution with the temperature T, we find the mean value of $|v|^{-1}$

$$\overline{|v|^{-1}} = (\tfrac{1}{2}\pi m/kT)^{-\frac{1}{2}}.$$

Hence the mean value of the time of flight from one point on the circumference to any other point on the circumference is

$$R\left(\frac{8}{\pi}\frac{m}{kT}\right)^{\frac{1}{2}}.$$

Let us use this as an estimate of d/v in our definition of s (equation 18 of § 2.2a). If we then take $T=\frac{1}{2}10^5\,°K$, $R=3\times10^{-3}$ m and $f=5\times10^9\,c/s$ as a suitable approximation in Dattner's experiment, we find $s=55$ and $\frac{1}{2}(s^{\frac{1}{2}}-1)\approx 27$. This last quantity should equal the number of resonances below the density making the plasma frequency equal to the frequency of measurement. The extrapolated experimental value is about 8.

In Romell's experiment fR is roughly the same as in Dattner's experiment and the observed number of resonances is 3. The extrapolated value of the number of resonances is here roughly 6.

Despite the fact that our theory is not developed for the cylindrical case nor accounts for a continuous velocity distribution, we obtain the same order of magnitude for the number of roots as can be extrapolated from the experiments. From this fact we cannot draw the conclusion that our theory explains the experiments. However, our results can be taken as an encouragement for further research along the lines followed here.

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